

# N-QUASI-ABELIAN CATEGORIES VS N-TILTING TORSION PAIRS

LUISA FIOROT

**ABSTRACT.** Rump in [Rum01] and successively Bondal and Van den Bergh [BvdB03] provide an equivalence between the notion of quasi-abelian category studied by Schneiders in [Sch99] and that of tilting torsion pair on an abelian category. Any 1-tilting object in an abelian category  $\mathcal{A}$  defines a tilting torsion pair on  $\mathcal{A}$ . We propose to extend the 1-picture to any  $n \geq 1$ . We propose a definition of  $n$ -quasi-abelian category, a definition of  $n$ -tilting torsion class in an abelian category, a procedure to tilt a  $t$ -structure  $\mathcal{D}$  with respect to a  $n$ -tilting torsion class on its heart  $\mathcal{H}_{\mathcal{D}}$ . Via the  $n$ -version of the Tilting Theorem proved in [FMS15] we prove that these notions are equivalent. In the hierarchy of  $n$ -quasi-abelian categories 0-quasi-abelian categories are abelian categories, 1-quasi-abelian categories are Schneiders quasi-abelian categories, 2-quasi-abelian categories are additive categories admitting kernels and cokernels. Any  $n$ -quasi-abelian category  $\mathcal{E}$  admits a “derived” category  $D(\mathcal{E})$  endowed with a  $n$ -tilting pair of  $t$ -structures  $(\mathcal{R}, \mathcal{L})$  such that  $\mathcal{E}$  coincides with the intersection of their hearts. Moreover we will provide an interpretation of the hearts of these  $t$ -structures in terms of coherent functors via the Yoneda embedding.

## INTRODUCTION

In 1982 Beilinson, Bernstein and Deligne, in their study of perverse sheaves on an algebraic or analytic variety, introduced in [BBD82] the notion of  $t$ -structure in a triangulated category. In particular they proved that the heart  $\mathcal{H}_{\mathcal{D}}$  of a  $t$ -structure  $\mathcal{D}$  in a triangulated category is an abelian category.

Soon after it has became clear that the notion of  $t$ -structure is the counterpart for triangulated category of the notion of torsion pair for an abelian category (see [BR07]). Moreover in 1996 Happel Reiten and Smalø in their work [HRS96] connected these two notions in a deeper way. They provided a “functorial” way to associate to any torsion pair  $(\mathcal{X}, \mathcal{Y})$  in the heart  $\mathcal{H}_{\mathcal{D}}$  of a  $t$ -structure  $\mathcal{D}$  on a triangulated category  $\mathcal{C}$  a new  $t$ -structure  $\mathcal{T}$  such that  $\mathcal{D}^{\leq -1} \subseteq \mathcal{T}^{\leq 0} \subseteq \mathcal{D}^{\leq 0}$  and moreover  $(\mathcal{Y}[1], \mathcal{X})$  provides a torsion pair in  $\mathcal{H}_{\mathcal{T}}$ . The  $t$ -structure  $\mathcal{T}$  is called the  $t$ -structure obtained by tilting  $\mathcal{D}$  with respect to the torsion pair  $(\mathcal{X}, \mathcal{Y})$ . In [Pol07] (2007) Polishchuk proved that fixed a  $t$ -structure  $\mathcal{D}$  on a triangulated category  $\mathcal{C}$  any  $t$ -structure  $\mathcal{T}$  such that  $\mathcal{D}^{\leq -1} \subseteq \mathcal{T}^{\leq 0} \subseteq \mathcal{D}^{\leq 0}$  is obtained by tilting  $\mathcal{D}$  with respect to the torsion pair  $(\mathcal{X} := \mathcal{H}_{\mathcal{D}} \cap \mathcal{T}^{\leq 0}, \mathcal{H}_{\mathcal{D}} \cap \mathcal{T}^{\geq 1} =: \mathcal{Y})$ .

Motivated by their study of quasi-tilted algebras Happel Reiten and Smalø proved in [HRS96] (and in its unbounded version by [Che10]) the so called Tilting Theorem: whenever  $(\mathcal{X}, \mathcal{Y})$  is a tilting torsion pair on an abelian category  $\mathcal{A}$  (i.e.;  $\mathcal{X}$  cogenerates  $\mathcal{A}$ ) the inclusion of the heart of the tilted  $t$ -structure  $\mathcal{H}_{\mathcal{T}} \hookrightarrow D(\mathcal{A})$  extends to a triangulated equivalence  $D(\mathcal{H}_{\mathcal{T}}) \cong D(\mathcal{A})$  and moreover  $(\mathcal{Y}[1], \mathcal{X})$  is a

co-tilting torsion pair in  $\mathcal{H}_{\mathcal{T}}$  (i.e.; the torsion class  $\mathcal{X}$  generates  $\mathcal{H}_{\mathcal{T}}$ ). Motivated by this result (following [FMT14] notation) a pair of  $t$ -structures  $(\mathcal{D}, \mathcal{T})$  on a triangulated category  $\mathcal{C}$  is called 1-tilting (respectively 1-cotilting) if  $\mathcal{D}^{\leq -1} \subseteq \mathcal{T}^{\leq 0} \subseteq \mathcal{D}^{\leq 0}$ ,  $\mathcal{C} \cong D(\mathcal{H}_{\mathcal{D}}) \cong D(\mathcal{H}_{\mathcal{T}})$  and  $\mathcal{E} := \mathcal{H}_{\mathcal{D}} \cap \mathcal{H}_{\mathcal{T}}$  is a tilting torsion class in  $\mathcal{H}_{\mathcal{D}}$  (respectively  $\mathcal{E}$  is a cotilting torsion-free class in  $\mathcal{H}_{\mathcal{T}}$ ).

In 1999 J.-P. Schneiders devoted his work [Sch99] to the study of 1-*quasi-abelian* categories (i.e. additive categories admitting kernels and cokernels and such that any kernel-cokernel short exact sequence is stable by push-out and pull-back). This notion seems to have been introduced in the sixties in [Jur] and we quote Rump paper [Rum08] for a short history of this notion. In [Sch99] Schneiders associated to any 1-quasi-abelian category  $\mathcal{E}$  a triangulated category  $D(\mathcal{E})$  endowed with a 1-tilting pair of  $t$ -structures  $(\mathcal{R}, \mathcal{L})$  such that  $\mathcal{E} = \mathcal{H}_{\mathcal{R}} \cap \mathcal{H}_{\mathcal{L}}$ .

Firstly Rump in [Rum01] and hence Bondal and Van den Bergh in their [BvdB03, Appendix B] provided (combining the above results) a bright equivalence between the previous notions: there is a one-to-one correspondence between quasi-abelian categories, tilting torsion (respectively cotilting torsion-free) classes, 1-tilting pairs of  $t$ -structures. In this equivalence a quasi-abelian category  $\mathcal{E}$  can be represented as a tilting torsion class  $\mathcal{E} \hookrightarrow \mathcal{H}_{\mathcal{R}}$  and as a co-tilting torsion-free class  $\mathcal{E} \hookrightarrow \mathcal{H}_{\mathcal{L}}$ .

The main aim of this paper is to extend the previous equivalence to the so called  $n$ -tilting case.

One can naturally generalize the  $t$ -structure side: a pair of  $t$ -structures  $(\mathcal{D}, \mathcal{T})$  on a triangulated category  $\mathcal{C}$  is called  $n$ -tilting if  $\mathcal{D}^{\leq -n} \subseteq \mathcal{T}^{\leq 0} \subseteq \mathcal{D}^{\leq 0}$  and  $\mathcal{C} \cong D(\mathcal{H}_{\mathcal{D}}) \cong D(\mathcal{H}_{\mathcal{T}})$ . In this direction Happel Reiten and Smalø Tilting Theorem has been recently extended to the so called  $n$ -tilting case in [FMT14] and [FMS15]. What remains undefined is the 1-quasi-abelian side and the notions of  $n$ -tilting torsion class (respectively  $n$ -cotilting torsion-free class).

We propose in this paper a definition of  $n$ -quasi-abelian category such that: 0-quasi-abelian categories are abelian categories, 1-quasi-abelian categories are Schneiders 1-quasi-abelian categories, 2-quasi-abelian categories are additive categories admitting kernels and cokernels. Moreover any  $n$ -quasi-abelian category  $\mathcal{E}$  has a derived category  $D(\mathcal{E})$  endowed with a  $n$ -tilting pair of  $t$ -structures  $(\mathcal{R}, \mathcal{L})$  such that  $\mathcal{E} = \mathcal{H}_{\mathcal{R}} \cap \mathcal{H}_{\mathcal{L}}$  which permits to extend the previous equivalence.

## 1. 1-QUASI-ABELIAN CATEGORIES VS 1-TILTING TORSION PAIRS

In the hierarchy of  $n$ -quasi-abelian categories, the 0-level corresponds to abelian categories while the 1-level corresponds to Schneiders notion of quasi-abelian category ([Sch99]) defined in the following way:

**Definition 1.1.** An additive category  $\mathcal{E}$  is called 1-*quasi-abelian* if it admits kernels and cokernels, and any push-out of a kernel is a kernel, and any pullback of a cokernel is a cokernel.

A zero sequence  $0 \longrightarrow E \xrightarrow{u} F \xrightarrow{v} G \longrightarrow 0$  is called *exact* if and only if  $(E, u)$  is the kernel of  $v$  and  $(G, v)$  is the cokernel of  $u$ ; hence the class of kernel-cokernel exact sequences provides the maximal Quillen exact structure on  $\mathcal{E}$  if and only if  $\mathcal{E}$  is 1-quasi-abelian (see Appendix A for the notion of maximal Quillen exact structure).

A complex  $X^\bullet$  with entries in  $\mathcal{E}$  is called *acyclic* if each differential  $d^n : X^n \rightarrow X^{n+1}$  decomposes in  $\mathcal{E}$  as  $d^n = m_n \circ e_n : X^n \xrightarrow{e_n} D^n \xrightarrow{m_n} X^{n+1}$  where  $m_n$  is the kernel of  $e_{n+1}$ , and  $e_{n+1}$  is the cokernel of  $m_n$  for any  $n \in \mathbb{Z}$ .

**1.2. Torsion pairs in abelian categories** ([Dic66]). A *torsion pair* in an abelian category  $\mathcal{A}$  is a pair  $(\mathcal{X}, \mathcal{Y})$  of full subcategories of  $\mathcal{A}$  satisfying the following conditions:

- (i)  $\text{Hom}_{\mathcal{A}}(X, Y) = 0$ , for every  $X \in \mathcal{X}$  and every  $Y \in \mathcal{Y}$ .
- (ii) For any object  $C \in \mathcal{A}$  there exists a short exact sequence in  $\mathcal{A}$

$$0 \rightarrow X \rightarrow C \rightarrow Y \rightarrow 0$$

with  $X \in \mathcal{X}$  and  $Y \in \mathcal{Y}$ .

Hence the “inclusion” functor  $i_{\mathcal{X}} : \mathcal{X} \rightarrow \mathcal{A}$  has a right adjoint  $\tau$  while  $i_{\mathcal{Y}} : \mathcal{Y} \rightarrow \mathcal{A}$  has a left adjoint  $\phi$ ; the endo-functors  $t := i_{\mathcal{X}} \circ \tau$  and  $f := i_{\mathcal{Y}} \circ \phi$  are radicals. Moreover  $\mathcal{X}$  (respectively  $\mathcal{Y}$ ) is closed under extensions, quotients (respectively subobjects) representable direct sums (respectively direct products).

**Remark 1.3.** As observed in [BvdB03, 5.4] both  $\mathcal{X}$  and  $\mathcal{Y}$  admits kernels and cokernels such that:  $\text{Ker}_{\mathcal{X}} = \tau \circ \text{Ker}_{\mathcal{A}}$ ,  $\text{Coker}_{\mathcal{X}} = \text{Coker}_{\mathcal{A}}$ ,  $\text{Ker}_{\mathcal{Y}} = \text{Ker}_{\mathcal{A}}$  and  $\text{Coker}_{\mathcal{Y}} = \phi \circ \text{Coker}_{\mathcal{A}}$ . Exact sequences in  $\mathcal{X}$  (respectively in  $\mathcal{Y}$ ) coincide with short exact sequences in  $\mathcal{A}$  whose terms belong to  $\mathcal{X}$  (respectively  $\mathcal{Y}$ ) and hence they are stable by pullbacks and push-out thus proving that  $\mathcal{X}$  and  $\mathcal{Y}$  are 1-quasi-abelian categories.

**Definition 1.4.** ([HRS96]) A torsion pair  $(\mathcal{X}, \mathcal{Y})$  is called *tilting* if  $\mathcal{X}$  *cogenerates*  $\mathcal{A}$  (i.e.; every object in  $\mathcal{A}$  is a subobject of an object in  $\mathcal{X}$ ) and  $\mathcal{X}$  is called a *1-tilting torsion class*. Dually  $(\mathcal{X}, \mathcal{Y})$  is *cotilting* if  $\mathcal{Y}$  *generates*  $\mathcal{A}$  (i.e.; every object in  $\mathcal{A}$  is a quotient of an object in  $\mathcal{Y}$ ) and  $\mathcal{Y}$  is called a *1-cotilting torsion-free class*.

**Lemma 1.5.** *The full subcategory  $\mathcal{E} \hookrightarrow \mathcal{A}$  is a 1-tilting torsion class if and only if*

- (1)  $\mathcal{E}$  *cogenerates*  $\mathcal{A}$ ;
- (2)  $\mathcal{E}$  *has kernels*;
- (3)  $\mathcal{E}$  *is closed under extensions in*  $\mathcal{A}$ ;
- (4) *for any exact sequence  $0 \rightarrow A \rightarrow X \rightarrow B \rightarrow 0$  in  $\mathcal{A}$  with  $X \in \mathcal{E}$  and  $A, B \in \mathcal{A}$  we have  $B \in \mathcal{E}$ .*

*Proof.* Any tilting torsion pair (see 1.2 and Definition 1.4) satisfies these 4 conditions. On the other side let  $\mathcal{E} \hookrightarrow \mathcal{A}$  be a full subcategory satisfying the previous conditions. Hence, by the first property, we can co-present any  $A \in \mathcal{A}$  as  $A = \text{Ker}_{\mathcal{A}} f$  with  $f : X_1 \rightarrow X_2$  and  $X_i \in \mathcal{E}$  for  $i = 1, 2$  and so, since the functor  $\text{Mod-}\mathcal{E} \ni \mathcal{A}(i(-), A) \cong \mathcal{E}(-, \text{Ker}_{\mathcal{E}} f)$ , we can define  $\tau(A) := \text{Ker}_{\mathcal{E}} f$  (using the second property) which gives a right adjoint of  $i$ . The fourth property implies that for any  $A \in \mathcal{A}$  the co-unit of the adjunction  $\varepsilon_A : i\tau(A) \rightarrow A$  is a monomorphism. So for any  $A \in \mathcal{A}$  we have a short exact sequence  $0 \rightarrow i\tau(A) \xrightarrow{\varepsilon_A} A \rightarrow \text{Coker}(\varepsilon_A) \rightarrow 0$ . Moreover  $\text{Coker}_{\mathcal{A}}(\varepsilon_A) \in \mathcal{E}^{\perp}$  (see Appendix C.1 for the notion of orthogonal class) since given any morphism  $f : E \rightarrow \text{Coker}_{\mathcal{A}}(\varepsilon_A)$  with  $E \in \mathcal{E}$  its  $\mathcal{A}$  pull-back  $A \times_{\text{Coker}_{\mathcal{A}}(\varepsilon_A)} E \in \mathcal{E}$  (by the third property since it is an extension of  $E$  by  $i\tau(A)$ ) and hence the pull-back morphism  $f' : A \times_{\text{Coker}_{\mathcal{A}}(\varepsilon_A)} E \rightarrow A$  factors (by adjunction) through  $i\tau(A)$  which implies that  $f = 0$ .  $\square$

We note that the torsion pair  $(\mathcal{A}, 0)$  in an abelian category  $\mathcal{A}$  is tilting while  $(0, \mathcal{A})$  is cotilting. So the identity  $\text{id} : \mathcal{A} \rightarrow \mathcal{A}$  represents  $\mathcal{A}$  as a tilting torsion class and also as a 1-cotilting torsion-free class.

**Example 1.6.** ([And09, Example 1.2.13]). In the following  $R$  is a commutative ring.

- (1) The category of torsion-free finitely generated modules over any domain  $R$  is 1-quasi-abelian. If  $R$  is Dedekind (or more generally Prufer), this is the category of projective modules of finite rank. If  $R$  is principal (or more generally Bézout), this is the category of free modules of finite rank.
- (2) The category of (finitely generated) reflexive modules over an integrally closed domain  $R$  is quasi-abelian. Kernel and cokernels in this category are the double duals of kernels and cokernels taken in the category of  $R$ -modules. If  $R$  is regular of dimension 2, this is the category of projective modules of finite rank.
- (3) The category of torsion-free coherent sheaves over a reduced irreducible analytic space or algebraic variety  $X$  is 1-quasi-abelian. If  $X$  is a normal curve, this is the category of vector bundles (of finite rank).
- (4) The category of filtered modules over any ring is 1-quasi-abelian.

Moreover there are many examples of 1-quasi-abelian categories from functional analysis such as various categories of topological vector spaces: Banach and Fréchet spaces, locally convex and nuclear spaces, bornological spaces of convex type.

We will refer to Appendix C for some generalities on  $t$ -structures. In particular in order to assure that any category introduced in this work has Hom sets we will suppose in the whole paper the following:

**1.7.** Given  $\mathcal{E}$  a *projectively complete* category (i.e. additive category such that any idempotent splits) its derived category  $D(\mathcal{E}) := D(\mathcal{E}, \mathcal{E}x_{\max})$  (endowed with its maximal Quillen exact structure see Appendix A) has Hom sets.

In the following we will always suppose that  $\mathcal{E}$  is a projectively complete category.

**1.8. Happel-Reiten-Smalø tilted  $t$ -structure.** [HRS96, Prop. I.2.1, Prop. I.3.2] [Bri05, Prop. 2.5]. Let  $\mathcal{H}_{\mathcal{D}}$  be the heart of a non degenerate  $t$ -structure  $\mathcal{D} = (\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0})$  on a triangulated category  $\mathcal{C}$  and let  $(\mathcal{X}, \mathcal{Y})$  be a torsion pair on  $\mathcal{H}_{\mathcal{D}}$ . Then the pair  $\mathcal{T} := (\mathcal{T}_{(\mathcal{X}, \mathcal{Y})}^{\leq 0}, \mathcal{T}_{(\mathcal{X}, \mathcal{Y})}^{\geq 0})$  of full subcategories of  $\mathcal{C}$

$$\begin{aligned} \mathcal{T}_{(\mathcal{X}, \mathcal{Y})}^{\leq 0} &= \{C \in \mathcal{C} \mid H_{\mathcal{D}}^0(C) \in \mathcal{X}, H_{\mathcal{D}}^i(C) = 0 \forall i > 0\} \\ \mathcal{T}_{(\mathcal{X}, \mathcal{Y})}^{\geq 0} &= \{C \in \mathcal{C} \mid H_{\mathcal{D}}^{-1}(C) \in \mathcal{Y}, H_{\mathcal{D}}^i(C) = 0 \forall i < -1\} \end{aligned}$$

is a  $t$ -structure on  $\mathcal{C}$ . One says that  $\mathcal{T}_{(\mathcal{X}, \mathcal{Y})}$  is obtained *by tilting  $\mathcal{D}$  with respect to the torsion pair  $(\mathcal{X}, \mathcal{Y})$* . Moreover  $(\mathcal{Y}[1], \mathcal{X})$  is a torsion pair in  $\mathcal{H}_{\mathcal{T}}$  called the *tilted torsion pair*.

In [Pol07, Lemma 1.1.2] Polishchuk proved that given any pair of  $t$ -structures  $(\mathcal{D}, \mathcal{T})$  on a triangulated category  $\mathcal{C}$  such that  $\mathcal{D}^{\leq -1} \subseteq \mathcal{T}^{\leq 0} \subseteq \mathcal{D}^{\leq 0}$ , the  $t$ -structure  $\mathcal{T}$  is obtained by tilting  $\mathcal{D}$  with respect to the torsion pair  $(\mathcal{X} := \mathcal{H}_{\mathcal{T}} \cap \mathcal{H}_{\mathcal{D}}, \mathcal{H}_{\mathcal{T}}[-1] \cap \mathcal{H}_{\mathcal{D}} =: \mathcal{Y})$  while  $\mathcal{D}$  is obtained by tilting  $\mathcal{T}$  with respect to the tilted torsion pair  $(\mathcal{Y}[1] = \mathcal{H}_{\mathcal{D}}[1] \cap \mathcal{H}_{\mathcal{T}}, \mathcal{H}_{\mathcal{D}} \cap \mathcal{H}_{\mathcal{T}} =: \mathcal{X})$ .

**Theorem 1.9. 1-tilting Theorem.** ([HRS96, Theorem I.3.3], [Che10]). *Given a tilting torsion pair  $(\mathcal{E}, \mathcal{Y})$  in  $\mathcal{A}$  there exists a triangle equivalence which extends the natural inclusion  $\mathcal{H}_{\mathcal{T}} \subseteq D(\mathcal{A})$*

$$\begin{array}{ccc} & \frac{K(\mathcal{E})}{\mathcal{N}} & \\ \cong \swarrow & & \searrow \cong \\ D(\mathcal{H}_{\mathcal{T}}) & \cdots \cdots \cdots \cong \cdots \cdots \cdots & D(\mathcal{A}) \end{array}$$

where  $\mathcal{N}$  is the null system of complexes in  $K(\mathcal{E})$  acyclic in  $\mathcal{A}$  or equivalently in  $\mathcal{H}_{\mathcal{T}}$ . Moreover  $(\mathcal{Y}[1], \mathcal{E})$  is a cotilting torsion pair in  $\mathcal{H}_{\mathcal{T}}$ .

**Definition 1.10.** A pair of  $t$ -structures  $(\mathcal{D}, \mathcal{T})$  on a triangulated category  $\mathcal{C}$  is called *1-tilting* if the following statements hold:

- (1)  $\mathcal{D}^{\leq -1} \subseteq \mathcal{T}^{\leq 0} \subseteq \mathcal{D}^{\leq 0}$ ;
- (2) denoted by  $\mathcal{E} := \mathcal{H}_{\mathcal{D}} \cap \mathcal{H}_{\mathcal{T}}$  the following equivalent conditions are satisfied:
  - (i):  $\mathcal{C} \cong D(\mathcal{H}_{\mathcal{D}}) \cong D(\mathcal{H}_{\mathcal{T}}) \cong K(\mathcal{E})/\mathcal{N}$  with  $\mathcal{E} := \mathcal{H}_{\mathcal{D}} \cap \mathcal{H}_{\mathcal{T}}$  (where  $\mathcal{N}$  is the null system of complexes in  $K(\mathcal{E})$  acyclic in  $\mathcal{H}_{\mathcal{D}}$  or equivalently in  $\mathcal{H}_{\mathcal{T}}$ );
  - (ii):  $\mathcal{C} \cong D(\mathcal{H}_{\mathcal{D}})$  and  $\mathcal{E}$  cogenerates  $\mathcal{H}_{\mathcal{D}}$ ;
  - (iii):  $\mathcal{C} \cong D(\mathcal{H}_{\mathcal{T}})$  and  $\mathcal{E}$  generates  $\mathcal{H}_{\mathcal{T}}$ .

**Proposition 1.11.** *The pair  $(\mathcal{D}, \mathcal{T})$  is a 1-tilting pair of  $t$ -structures if and only if  $\mathcal{E} := \mathcal{H}_{\mathcal{D}} \cap \mathcal{H}_{\mathcal{T}}$  is a tilting torsion class (respectively tilting torsion-free class) in  $\mathcal{H}_{\mathcal{D}}$  (respectively in  $\mathcal{H}_{\mathcal{T}}$ ).*

*Proof.* By the Tilting Theorem 1.9 if  $\mathcal{E} := \mathcal{H}_{\mathcal{D}} \cap \mathcal{H}_{\mathcal{T}}$  is a tilting torsion class (respectively tilting torsion-free class) in  $\mathcal{H}_{\mathcal{D}}$  (respectively in  $\mathcal{H}_{\mathcal{T}}$ ) we obtain that  $(\mathcal{D}, \mathcal{T})$  is a 1-tilting pair of  $t$ -structures. On the other side if  $\mathcal{C} \cong D(\mathcal{H}_{\mathcal{D}}) \cong K(\mathcal{E})/\mathcal{N}$  we have that  $\mathcal{E}$  cogenerates  $\mathcal{H}_{\mathcal{D}}$  since any  $A \in \mathcal{H}_{\mathcal{D}}$  can be represented by a complex  $E^\bullet \in K(\mathcal{E})$  and so  $A \hookrightarrow \text{Coker}_{\mathcal{H}_{\mathcal{D}}}(d_{E^\bullet}^{-1}) \in \mathcal{E}$  (and  $\text{Coker}_{\mathcal{H}_{\mathcal{D}}}(d_{E^\bullet}^{-1}) \in \mathcal{E}$  since it is a quotient of a torsion object in  $\mathcal{H}_{\mathcal{D}}$ ) and it is a torsion class in  $\mathcal{H}_{\mathcal{D}}$  by [Pol07, Lemma 1.1.2]. Dually if  $\mathcal{C} \cong D(\mathcal{H}_{\mathcal{T}}) \cong K(\mathcal{E})/\mathcal{N}$  we have that  $\mathcal{E}$  generates  $\mathcal{H}_{\mathcal{T}}$  and it is a torsion-free class in  $\mathcal{H}_{\mathcal{T}}$ .  $\square$

**1.12. Left and Right  $t$ -structures on the derived category of a quasi-abelian category** ([Sch99, §1.2]). Let  $K^{\leq 0}(\mathcal{E})$  (respectively  $K^{\geq 0}(\mathcal{E})$ ) denote the full subcategory of  $K(\mathcal{E})$  formed by complexes which are isomorphic in  $K(\mathcal{E})$  to complexes whose entries in each strictly positive (respectively strictly negative) degree are zero. Let now suppose that  $\mathcal{E}$  admits kernels and cokernels, hence the pairs  $\mathcal{LK}_{\mathcal{E}} := (K^{\leq 0}(\mathcal{E}), (K^{\leq -1}(\mathcal{E}))^\perp)$  and  $\mathcal{RK}_{\mathcal{E}} := ({}^\perp(K^{\geq 1}(\mathcal{E})), K^{\geq 0}(\mathcal{E}))$  define two  $t$ -structures on  $K(\mathcal{E})$  whose truncation functors are respectively:

$$\begin{aligned} \tau_{\mathcal{L}}^{\leq 0} E^\bullet &:= \cdots \longrightarrow E^{-2} \longrightarrow E^{-1} \longrightarrow \overset{\bullet}{\text{Ker}}_{\mathcal{E}} d^0 \longrightarrow 0 \longrightarrow \cdots \\ \tau_{\mathcal{L}}^{\geq 1} E^\bullet &:= \cdots \longrightarrow 0 \longrightarrow \text{Ker}_{\mathcal{E}} d^0 \longrightarrow \overset{\bullet}{E}^0 \longrightarrow E^1 \longrightarrow \cdots \\ \tau_{\mathcal{R}}^{\leq -1} E^\bullet &:= \cdots \longrightarrow E^{-1} \longrightarrow \overset{\bullet}{E}^0 \longrightarrow \text{Coker}_{\mathcal{E}} d^{-1} \longrightarrow 0 \longrightarrow \cdots \\ \tau_{\mathcal{R}}^{\geq 0} E^\bullet &:= \cdots \longrightarrow 0 \longrightarrow \text{Coker}_{\mathcal{E}} d^{-1} \longrightarrow E^1 \longrightarrow E^2 \longrightarrow \cdots \end{aligned}$$

(as in C.5 we use a point to indicate the object placed in degree 0). We will denote by  $\mathcal{LK}(\mathcal{E})$  (respectively  $\mathcal{RK}(\mathcal{E})$ ) the heart associated to the  $t$ -structure  $\mathcal{LK}_{\mathcal{E}}$  (respectively  $\mathcal{RK}_{\mathcal{E}}$ ). We have  $\mathcal{E} = K^{\leq 0}(\mathcal{E}) \cap K^{\geq 0}(\mathcal{E}) = \mathcal{LK}(\mathcal{E}) \cap \mathcal{RK}(\mathcal{E})$  in  $K(\mathcal{E})$  and moreover  $\mathcal{RK}_{\mathcal{E}}^{\leq -2} \subseteq \mathcal{LK}_{\mathcal{E}}^{\leq 0} \subseteq \mathcal{RK}_{\mathcal{E}}^{\leq 0}$  (since for any  $E^\bullet \in K(\mathcal{E})$  its  $\tau_{\mathcal{R}}^{\leq -2}(E^\bullet) \in K^{\leq 0}(\mathcal{E})$ ). In  $K(\mathcal{E})$  we have that  $\mathcal{RK}_{\mathcal{E}}^{\leq -1}$  is contained in  $\mathcal{LK}_{\mathcal{E}}^{\leq 0}$  if and only if any cokernel map is a split epimorphism or equivalently any kernel map is a split monomorphism. If this is not the case in order to reduce the “gap” ([FMT14, Definition 2.1]) between the left and the right  $t$ -structures (without changing the intersection  $\mathcal{E}$ ) we can try to localize by a null system formed by acyclic complexes with respect to a Quillen exact structure (see Appendix A.5). In this case, if the previous  $t$ -structures satisfy the conditions of Lemma C.12, they will induce

a pair of  $t$ -structures  $(\mathcal{RD}_{(\mathcal{E}, \mathcal{E}x)}, \mathcal{LD}_{(\mathcal{E}, \mathcal{E}x)})$  on  $D(\mathcal{E}, \mathcal{E}x)$ . In order to have that  $\mathcal{RD}_{(\mathcal{E}, \mathcal{E}x)}^{\leq -1} \subseteq \mathcal{LD}_{(\mathcal{E}, \mathcal{E}x)}^{\leq 0} \subseteq \mathcal{RD}_{(\mathcal{E}, \mathcal{E}x)}^{\leq 0}$  we need to prove that for any  $E^\bullet \in D(\mathcal{E})$  the canonical morphism of complexes  $\alpha_{E^\bullet} : \tau_{\mathcal{L}}^{\leq 0}(\tau_{\mathcal{R}}^{\leq -1} E^\bullet) \rightarrow \tau_{\mathcal{R}}^{\leq -1} E^\bullet$  is an isomorphism in  $D(\mathcal{E}, \mathcal{E}x)$ :

$$\begin{array}{ccccccc} \tau_{\mathcal{L}}^{\leq 0}(\tau_{\mathcal{R}}^{\leq -1} E^\bullet) & := & \cdots & \longrightarrow & E^{-1} & \longrightarrow & \text{Im}_{\mathcal{E}}(d^{-1}) \longrightarrow 0 \longrightarrow 0 \longrightarrow \cdots \\ \alpha_{E^\bullet} \downarrow & & & & \downarrow & & \downarrow \\ \tau_{\mathcal{R}}^{\leq -1} E^\bullet & := & \cdots & \longrightarrow & E^{-1} & \xrightarrow{d^{-1}} & E^0 \longrightarrow \text{Coker}_{\mathcal{E}}(d^{-1}) \longrightarrow 0 \longrightarrow \cdots \end{array}$$

This is equivalent to the acyclicity of the mapping cone  $M(\alpha_{E^\bullet})$  which is homotopically isomorphic to  $\text{Ex}(d^0)$ :

$$\begin{array}{ccccccc} M(\alpha_{E^\bullet}) & := & \cdots & \longrightarrow & E^{-2} \oplus E^{-1} & \longrightarrow & E^{-1} \oplus \text{Im}_{\mathcal{E}}(d^{-1}) \longrightarrow \dot{E}^0 \longrightarrow \text{Coker}_{\mathcal{E}}(d^{-1}) \longrightarrow 0 \longrightarrow \cdots \\ \cong \downarrow & & & & \downarrow & & \downarrow \\ \text{Ex}(d^0) & := & \cdots & \longrightarrow & 0 & \longrightarrow & \text{Im}_{\mathcal{E}}(d^{-1}) \longrightarrow \dot{E}^0 \longrightarrow \text{Coker}_{\mathcal{E}}(d^{-1}) \longrightarrow 0 \longrightarrow \cdots \end{array}$$

When  $\mathcal{E}$  is a 1-quasi-abelian category the previous truncation functors induce, by [Sch99, Lemma 1.2.17; 1.18] (see Proposition C.12 and Lemma 3.10), the  $t$ -structure  $\mathcal{LD}_{\mathcal{E}}$  (respectively  $\mathcal{RD}_{\mathcal{E}}$ ) in the derived category  $D(\mathcal{E}) = K(\mathcal{E})/\mathcal{N}$ . Moreover since the sequence  $0 \rightarrow \text{Im}_{\mathcal{E}}(d^{-1}) \rightarrow E^0 \rightarrow \text{Coker}_{\mathcal{E}}(d^{-1}) \rightarrow 0$  (which is a kernel-cokernel exact sequence) is exact for the maximal Quillen exact structure on  $\mathcal{E}$  we deduce  $\mathcal{RD}_{\mathcal{E}}^{\leq -1} \subseteq \mathcal{LD}_{\mathcal{E}}^{\leq 0} \subseteq \mathcal{RD}_{\mathcal{E}}^{\leq 0}$  and  $\mathcal{E} = \mathcal{LD}_{\mathcal{E}}^{\leq 0} \cap \mathcal{RD}_{\mathcal{E}}^{\geq 0}$ . The  $t$ -structure  $\mathcal{LD}_{\mathcal{E}}$  (respectively  $\mathcal{RD}_{\mathcal{E}}$ ) is called the *left  $t$ -structure* (respectively the *right  $t$ -structure*), whose aisle  $\mathcal{LD}_{\mathcal{E}}^{\leq 0}$  (respectively co-aisle  $\mathcal{RD}_{\mathcal{E}}^{\geq 0}$ ) is the class of complexes isomorphic in  $D(\mathcal{E})$  to complexes whose entries in each strictly positive (respectively negative) degree are zero. The heart of  $\mathcal{LD}_{\mathcal{E}}$  (respectively  $\mathcal{RD}_{\mathcal{E}}$ ) is denoted by  $\mathcal{LH}(\mathcal{E})$  (respectively  $\mathcal{RH}(\mathcal{E})$ ) and we denote by  $I_{\mathcal{L}}$  (respectively  $I_{\mathcal{R}}$ ) the canonical embedding into  $\mathcal{LH}(\mathcal{E})$  (respectively  $\mathcal{RH}(\mathcal{E})$ )

$$\begin{array}{ccc} I_{\mathcal{L}} : \mathcal{E} & \longrightarrow & \mathcal{LH}(\mathcal{E}) \\ E & \longmapsto & 0 \rightarrow \dot{E} \end{array} \qquad \begin{array}{ccc} I_{\mathcal{R}} : \mathcal{E} & \longrightarrow & \mathcal{RH}(\mathcal{E}) \\ E & \longmapsto & \dot{E} \rightarrow 0 \end{array}$$

which preserves and reflects exact sequences. Moreover  $\mathcal{E}$  is stable under extensions in  $\mathcal{LH}(\mathcal{E})$  (respectively  $\mathcal{RH}(\mathcal{E})$ ).

**Proposition 1.13.** *Let  $\mathcal{E}$  be a 1-quasi-abelian category. The  $t$ -structures  $\mathcal{LD}_{\mathcal{E}} = \mathcal{RD}_{\mathcal{E}}$  coincide if and only if  $\mathcal{E}$  is an abelian category.*

*Proof.* We note that if it was  $\mathcal{RD}_{\mathcal{E}} = \mathcal{LD}_{\mathcal{E}}$  hence  $\mathcal{E} \cong \mathcal{RH}(\mathcal{E}) \cong \mathcal{LH}(\mathcal{E})$  would be an abelian category. On the other side if  $\mathcal{E}$  is an abelian category it is also 1-quasi-abelian and so for any complex  $E^\bullet \in D(\mathcal{E})$  we have  $\tau_{\mathcal{R}}^{\leq 0} E^\bullet \cong \tau_{\mathcal{L}}^{\leq -1} \tau_{\mathcal{R}}^{\leq 0} E^\bullet$ , hence the

canonical map  $\beta_{E^\bullet} : \tau_{\mathcal{L}}^{\leq 0} E^\bullet \rightarrow \tau_{\mathcal{R}}^{\leq 0} E^\bullet \cong \tau_{\mathcal{L}}^{\leq 1} \tau_{\mathcal{R}}^{\leq 0} E^\bullet$

$$\begin{array}{ccccccc} \tau_{\mathcal{L}}^{\leq 0} E^\bullet := \cdots & \longrightarrow & E^{-1} & \longrightarrow & \text{Ker}_{\mathcal{E}} d^0 & \longrightarrow & 0 \longrightarrow 0 \longrightarrow \cdots \\ \beta_{E^\bullet} \downarrow & & \downarrow & & \downarrow & & \downarrow \\ \tau_{\mathcal{L}}^{\leq 1} \tau_{\mathcal{R}}^{\leq 0} E^\bullet = \cdots & \longrightarrow & E^{-1} & \xrightarrow{d^{-1}} & E^0 & \longrightarrow & \text{Im}_{\mathcal{E}}(d^0) \longrightarrow 0 \longrightarrow \cdots \end{array}$$

is an isomorphism in  $D(\mathcal{E})$  if and only if the short sequence  $0 \rightarrow \text{Ker}_{\mathcal{E}} d^0 \rightarrow E^0 \rightarrow \text{Im}_{\mathcal{E}}(d^0) \rightarrow 0$  is exact for the maximal Quillen exact structure on  $\mathcal{E}$  which is true if and only if  $\mathcal{E}$  is an abelian category.  $\square$

As proved by Bondal and Van den Bergh in [BvdB03, Prop. B.3] we can summarize the previous results of Schneiders [Sch99, 1.2.3] as follows:

**Theorem 1.14.** *There is a one-to-one correspondence between the classes*

$$\begin{array}{ccc} \{1\text{-quasi-abelian categories}\} & \longleftrightarrow & \{1\text{-tilting pairs of } t\text{-structures}\} \\ \uparrow & & \uparrow \\ \mathcal{E} = \mathcal{RH}(\mathcal{E}) \cap \mathcal{LH}(\mathcal{E}) & \longleftrightarrow & (\mathcal{RD}_{\mathcal{E}}, \mathcal{LD}_{\mathcal{E}}) \text{ on } \mathcal{C} = D(\mathcal{E}) \\ \downarrow & & \downarrow \\ \{1\text{-tilting torsion classes}\} & \longleftrightarrow & \{1\text{-cotilting torsion-free classes}\} \\ \downarrow & & \downarrow \\ \mathcal{E} \text{ in } \mathcal{RH}(\mathcal{E}) & \longleftrightarrow & \mathcal{E} \text{ in } \mathcal{LH}(\mathcal{E}). \end{array}$$

*Proof.* The equivalence between 1-tilting torsion classes (respectively 1-cotilting torsion-free classes) and 1-tilting pairs of  $t$ -structures is a consequence of the 1-Tilting Theorem 1.9 and Proposition 1.11. Given  $\mathcal{E}$  a 1-quasi-abelian category as recovered in 1.12 Schneiders proved that  $(\mathcal{RD}_{\mathcal{E}}, \mathcal{LD}_{\mathcal{E}})$  is a 1-tilting pair of  $t$ -structures with  $\mathcal{LH}(\mathcal{E}) \cap \mathcal{RH}(\mathcal{E}) = \mathcal{E}$ . On the other direction given any 1-tilting pair of  $t$ -structures  $(\mathcal{D}, \mathcal{T})$  by Proposition 1.11 the class  $\mathcal{E} := \mathcal{H}_{\mathcal{T}} \cap \mathcal{H}_{\mathcal{D}}$  is a tilting torsion class in  $\mathcal{H}_{\mathcal{D}}$  and hence a 1-quasi-abelian category.  $\square$

We have seen in 1.2 that given any torsion pair  $(\mathcal{X}, \mathcal{Y})$  in an abelian category  $\mathcal{A}$  both  $\mathcal{X}$  and  $\mathcal{Y}$  are 1-quasi-abelian categories, so in particular  $\mathcal{X}$  is a tilting torsion class after a suitable replacement of the abelian category:

**Proposition 1.15.** *Let  $(\mathcal{X}, \mathcal{Y})$  be any torsion pair in an abelian category  $\mathcal{A}$ . Let consider  $\mathcal{A}_{\mathcal{X}}$  to be the full subcategory of  $\mathcal{A}$  whose objects are cogenerated by  $\mathcal{X}$ . Then  $\mathcal{A}_{\mathcal{X}}$  is abelian, the canonical embedding functor  $\mathcal{A}_{\mathcal{X}} \rightarrow \mathcal{A}$  is exact and  $(\mathcal{X}, \mathcal{Y} \cap \mathcal{A}_{\mathcal{X}})$  is a tilting torsion pair in  $\mathcal{A}_{\mathcal{X}}$  therefore  $\mathcal{A}_{\mathcal{X}} \cong \mathcal{RH}(\mathcal{X})$ .*

*Dually let consider  $\mathcal{A}_{\mathcal{Y}}$  to be the full subcategory of  $\mathcal{A}$  whose objects are generated by  $\mathcal{Y}$ . Then  $\mathcal{A}_{\mathcal{Y}}$  is abelian, the functor  $\mathcal{A}_{\mathcal{Y}} \rightarrow \mathcal{A}$  is exact and  $(\mathcal{X} \cap \mathcal{A}_{\mathcal{Y}}, \mathcal{Y})$  is a cotilting torsion pair in  $\mathcal{A}_{\mathcal{Y}}$  therefore  $\mathcal{A}_{\mathcal{Y}} \cong \mathcal{LH}(\mathcal{Y})$ .*

*Proof.* Let us prove that for any  $X \xrightarrow{f} Y$  morphism in  $\mathcal{A}_{\mathcal{X}}$ , its kernel and cokernel in  $\mathcal{A}$  belong to  $\mathcal{A}_{\mathcal{X}}$ . By definition of  $\mathcal{A}_{\mathcal{X}}$  there exist  $X \xrightarrow{\alpha_X} T_X$  and  $Y \xrightarrow{\alpha_Y} T_Y$  with  $T_X, T_Y$  in  $\mathcal{X}$ . Hence  $\text{Ker}_{\mathcal{A}}(f) \hookrightarrow X \xrightarrow{\alpha_X} T_X$  implies  $\text{Ker}_{\mathcal{A}}(f) \in \mathcal{A}_{\mathcal{X}}$  while  $\text{Coker}_{\mathcal{A}}(f) \hookrightarrow \text{Coker}_{\mathcal{A}}(\alpha_Y f) \in \mathcal{X}$ , since  $\mathcal{X}$  is closed under quotients and  $T_Y \in \mathcal{X}$ . Let  $X \in \mathcal{A}_{\mathcal{X}}$  and let consider its short exact sequence  $0 \rightarrow T(X) \rightarrow X \rightarrow F(X) \rightarrow 0$  where  $T(X)$  (respectively  $F(X)$ ) is its torsion (respectively torsion-free) part with



respect to the torsion pair  $(\mathcal{X}, \mathcal{Y})$  in  $\mathcal{A}$ . Then  $T(X) \in \mathcal{X} \subseteq \mathcal{A}_{\mathcal{X}}$  and hence  $F(X) \in \mathcal{A}_{\mathcal{X}}$  (since it is a cokernel of a morphism in  $\mathcal{A}_{\mathcal{X}}$ ) which proves that  $(\mathcal{X}, \mathcal{Y} \cap \mathcal{A}_{\mathcal{X}})$  is a torsion pair in  $\mathcal{A}_{\mathcal{X}}$ . The second statement follows dually.  $\square$

## 2. $n$ -TILTING THEOREM

Theorem 1.14 provides a bridge between four different classes: 1-quasi-abelian categories, 1-tilting pairs of  $t$ -structures, tilting torsion classes, cotilting torsion-free classes. The main tools we used in Theorem 1.14 were Happel Reiten and Smalø 1-Tilting Theorem 1.9 and the procedure which permits to tilt a  $t$ -structure  $\mathcal{D}$  with respect to a torsion pair on  $\mathcal{H}_{\mathcal{D}}$ . The Tilting Theorem has recently been extended to the  $n$ -tilting case in [FMS15].

**2.1.** Let  $\mathcal{C}$  be a triangulated category endowed with a pair of  $t$ -structures  $(\mathcal{D}, \mathcal{T})$  such that  $\mathcal{D}^{\leq -n} \subseteq \mathcal{T}^{\leq 0} \subseteq \mathcal{D}^{\leq 0}$ . Let  $\mathcal{E} := \mathcal{H}_{\mathcal{T}} \cap \mathcal{H}_{\mathcal{D}}$ . The following statements hold true:

- (1) any complex  $\cdots \rightarrow 0 \rightarrow E^{-s} \rightarrow \cdots \rightarrow E^{-1} \rightarrow \overset{\bullet}{E^0} \rightarrow 0 \rightarrow \cdots$  with  $s \geq 0$  belongs to  $\mathcal{T}^{[-s, 0]} \cap \mathcal{D}^{[-s, 0]}$  ([FMS15, Lemma 1.1]);
- (2) if  $n \geq 1$  given an exact sequence in  $\mathcal{H}_{\mathcal{D}}$  (respectively  $\mathcal{H}_{\mathcal{T}}$ )

$$0 \longrightarrow M \xrightarrow{g} E_{-n+1} \xrightarrow{d_E^{-n+1}} \cdots \xrightarrow{d_E^{-1}} E_0 \xrightarrow{f} N \longrightarrow 0$$

with  $E_{-i} \in \mathcal{E}$  for any  $i = 0, \dots, n-1$  implies  $N = \text{Coker}_{\mathcal{H}_{\mathcal{D}}} d_E^{-1} \in \mathcal{E}$  (respectively  $M = \text{Ker}_{\mathcal{H}_{\mathcal{T}}} d_E^{-n+1} \in \mathcal{E}$ ) see [FMS15, Lemma 1.2];

- (3) a complex  $E^\bullet \in K(\mathcal{E})$  is acyclic in  $\mathcal{H}_{\mathcal{D}}$  if and only if it is acyclic in  $\mathcal{H}_{\mathcal{T}}$  and in this case for any differential  $\text{Ker}_{\mathcal{H}_{\mathcal{D}}} d_E^i \cong \text{Ker}_{\mathcal{H}_{\mathcal{T}}} d_E^i \in \mathcal{E}$  ([FMS15, Proposition 1.3]);
- (4)  $\mathcal{E}$  is projectively complete (any idempotent in  $\mathcal{E}$  splits in  $\mathcal{H}_{\mathcal{D}}$  and it belongs to  $\mathcal{H}_{\mathcal{T}}$  too);  $\mathcal{E}$  is closed under extensions both in  $\mathcal{H}_{\mathcal{D}}$  and  $\mathcal{H}_{\mathcal{T}}$  and hence the class of short exact sequences  $0 \rightarrow E_1 \rightarrow E \rightarrow E_2 \rightarrow 0$  (in  $\mathcal{H}_{\mathcal{D}}$  or equivalently  $\mathcal{H}_{\mathcal{T}}$ ) form a Quillen exact structure  $(\mathcal{E}, \mathcal{E}x)$  on  $\mathcal{E}$ .

**Remark 2.2.** Let  $\mathcal{C} = D(\mathcal{H}_{\mathcal{D}})$  and let us suppose that  $\mathcal{E}$  is cogenerating in  $\mathcal{H}_{\mathcal{D}}$ . By [FMS15, Lemma 1.4]  $\mathcal{E}$  is generating in  $\mathcal{H}_{\mathcal{T}}$  and by point (2) of 2.1 any  $A \in \mathcal{H}_{\mathcal{D}}$  admits a copresentation of length at most  $n$ . Dually any  $B \in \mathcal{H}_{\mathcal{T}}$  has a presentation of length at most  $n$ .

All the previous results combine into the following  $n$  version of Theorem 1.9

**Theorem 2.3.  $n$ -Tilting Theorem.** ([FMS15, Theorem 1.5] ) *Let  $\mathcal{A}$  be abelian category such that its derived category  $D(\mathcal{A})$  has Hom sets, let  $\mathcal{D}$  be the natural  $t$ -structure in  $D(\mathcal{A})$  and  $\mathcal{T}$  a  $t$ -structure such that  $\mathcal{D}^{\leq -n} \subseteq \mathcal{T}^{\leq 0} \subseteq \mathcal{D}^{\leq 0}$ . Let us suppose that  $\mathcal{E} := \mathcal{A} \cap \mathcal{H}_{\mathcal{T}}$  cogenerates  $\mathcal{A}$ , hence there exists a triangle equivalence*

$$\begin{array}{ccc} & \frac{K(\mathcal{E})}{\mathcal{N}_{\mathcal{E}x}} & \\ \cong \swarrow & & \searrow \cong \\ D(\mathcal{H}_{\mathcal{T}}) & \xrightarrow{\quad \cong \quad} & D(\mathcal{A}) \end{array}$$

(where  $\mathcal{N}_{\mathcal{E}x}$  is the null system of complexes in  $K(\mathcal{E})$  acyclic in  $\mathcal{A}$  or equivalently in  $\mathcal{H}_{\mathcal{T}}$ ) which extends the natural inclusion  $\mathcal{H}_{\mathcal{T}} \subseteq D(\mathcal{A})$ . Moreover  $\mathcal{E}$  is generating in  $\mathcal{H}_{\mathcal{T}}$ .



This theorem permits to introduce the notion of  $n$ -tilting pair of  $t$ -structures generalizing Definition 1.10

**Definition 2.4.** A pair of  $t$ -structures  $(\mathcal{D}, \mathcal{T})$  on a triangulated category  $\mathcal{C}$  is called  $n$ -tilting if the following statements hold:

- (1)  $\mathcal{D}^{\leq -n} \subseteq \mathcal{T}^{\leq 0} \subseteq \mathcal{D}^{\leq 0}$ ;
- (2) the following equivalent conditions are satisfied:
  - (i):  $\mathcal{C} \cong D(\mathcal{H}_{\mathcal{D}}) \cong D(\mathcal{H}_{\mathcal{T}}) \cong K(\mathcal{E})/\mathcal{N}_{\mathcal{E}x}$  with  $\mathcal{E} := \mathcal{H}_{\mathcal{D}} \cap \mathcal{H}_{\mathcal{T}}$  (where  $\mathcal{N}$  is the null system of complexes in  $K(\mathcal{E})$  acyclic in  $\mathcal{H}_{\mathcal{D}}$  or equivalently in  $\mathcal{H}_{\mathcal{T}}$ );
  - (ii):  $\mathcal{C} \cong D(\mathcal{H}_{\mathcal{D}})$  and  $\mathcal{E}$  cogenerates  $\mathcal{H}_{\mathcal{D}}$ ;
  - (iii):  $\mathcal{C} \cong D(\mathcal{H}_{\mathcal{T}})$  and  $\mathcal{E}$  generates  $\mathcal{H}_{\mathcal{T}}$ .

If  $\mathcal{D}^{\leq -n} \subseteq \mathcal{T}^{\leq 0} \subseteq \mathcal{D}^{\leq 0}$  by Theorem 2.3 we have that (ii) implies (i) and (iii), dually (iii) implies (i) and (ii) so (ii) is equivalent to (iii). If (i) holds hence  $\mathcal{E}$  cogenerates  $\mathcal{H}_{\mathcal{D}}$  since any  $A \in \mathcal{H}_{\mathcal{D}}$  can be represented by a complex  $E^\bullet \in K(\mathcal{E})$  and so  $A \hookrightarrow \text{Coker}_{\mathcal{H}_{\mathcal{D}}}(d_{E^\bullet}^{-1}) \in \mathcal{E}$  ( $\text{Coker}_{\mathcal{H}_{\mathcal{D}}}(d_{E^\bullet}^{-1}) \in \mathcal{E}$  by (2) of 2.1).

We note that, by definition, any  $n$ -tilting pair of  $t$ -structures is also  $m$ -tilting for any  $m \geq n$ .

**Proposition 2.5.** Let  $(\mathcal{D}, \mathcal{T})$  be a  $n$ -tilting pair of  $t$ -structures in  $K(\mathcal{E})/\mathcal{N}_{\mathcal{E}x} = D(\mathcal{E}, \mathcal{E}x)$  (where the Quillen exact structure on  $\mathcal{E}$  is the one of 2.1 (4) induced by short exact sequences in  $\mathcal{H}_{\mathcal{D}}$  or equivalently  $\mathcal{H}_{\mathcal{T}}$  whose terms belong to  $\mathcal{E}$ ). Hence

$$\mathcal{T}^{\leq 0} \cong \{X^\bullet \in K(\mathcal{E}) \mid X^\bullet \cong E_{\leq 0}^\bullet \text{ in } K(\mathcal{E})/\mathcal{N}_{\mathcal{E}x} \text{ with } E_{\leq 0}^\bullet \in K^{\leq 0}(\mathcal{E})\} =: \mathcal{LD}_{(\mathcal{E}, \mathcal{E}x)}^{\leq 0}$$

while

$$\mathcal{D}^{\geq 1} = \{X^\bullet \in K(\mathcal{E}) \mid X^\bullet \cong E_{\geq 1}^\bullet \text{ in } K(\mathcal{E})/\mathcal{N}_{\mathcal{E}x} \text{ with } E_{\geq 1}^\bullet \in K^{\geq 1}(\mathcal{E})\} =: \mathcal{RD}_{(\mathcal{E}, \mathcal{E}x)}^{\geq 1}.$$

*Proof.* By definition  $D(\mathcal{E}, \mathcal{E}x) = \frac{K(\mathcal{E})}{\mathcal{N}_{\mathcal{E}x}}$  (as introduced by Neeman in [Nee90] see Appendix A.5). Since  $(\mathcal{D}, \mathcal{T})$  is  $n$ -tilting we have that  $\mathcal{T}^{\leq 0} \cong D^{\leq 0}(\mathcal{H}_{\mathcal{T}})$  and  $\mathcal{D}^{\geq 1} \cong D^{\geq 1}(\mathcal{H}_{\mathcal{D}})$ . Moreover the class  $\mathcal{E}$  generates  $\mathcal{H}_{\mathcal{T}}$  and so any object in  $D^{\leq 0}(\mathcal{H}_{\mathcal{T}})$  can be represented in  $K(\mathcal{E})/\mathcal{N}_{\mathcal{E}x}$  by a complex in  $K^{\leq 0}(\mathcal{E})$ . On the other side since  $\mathcal{E}$  cogenerates  $\mathcal{H}_{\mathcal{D}}$  any object in  $D^{\geq 1}(\mathcal{H}_{\mathcal{D}})$  can be represented in  $K(\mathcal{E})/\mathcal{N}_{\mathcal{E}x}$  by a complex in  $K^{\geq 1}(\mathcal{E})$ .  $\square$

**Remark 2.6.** The proof of Theorem 2.3 produces the desired equivalence on the derived categories of the hearts passing through an equivalence with the triangulated category  $\frac{K(\mathcal{E})}{\mathcal{N}_{\mathcal{E}x}} = D(\mathcal{E}, \mathcal{E}x)$  where  $\mathcal{E}$  is the intersection of the hearts. The previous proposition proves that the category  $\mathcal{E}$  encodes the data of the  $t$ -structures since  $(\mathcal{D}, \mathcal{T}) \cong (\mathcal{RD}_{(\mathcal{E}, \mathcal{E}x)}, \mathcal{LD}_{(\mathcal{E}, \mathcal{E}x)})$ .

### 3. 2-QUASI-ABELIAN CATEGORIES VS 2-TILTING TORSION PAIRS

As we will see soon the case  $n = 2$  is neatly easier than  $n > 2$  and so we will first analyze this case in detail. We will proceed as follows: in Lemma 3.1 we will deduce, starting from a 2-tilting pair of  $t$ -structures  $(\mathcal{D}, \mathcal{T})$ , the properties of  $\mathcal{E} := \mathcal{H}_{\mathcal{D}} \cap \mathcal{H}_{\mathcal{T}}$ . Hence we will propose (in Definition 3.2) the definition of 2-quasi-abelian category and (in Definition 3.3) the notion of 2-tilting torsion (respectively 2-tilting torsion-free) class into an abelian category (which are 2-quasi-abelian categories). We will prove in Proposition 3.5 that any 2-tilting pair of  $t$ -structures  $(\mathcal{D}, \mathcal{T})$  permits to realize  $\mathcal{E} = \mathcal{H}_{\mathcal{D}} \cap \mathcal{H}_{\mathcal{T}}$  as a 2-tilting torsion class in  $\mathcal{H}_{\mathcal{D}}$  (respectively a 2-cotilting

torsion-free class in  $\mathcal{H}_{\mathcal{T}}$ ). While in Theorem 3.6 we will show how, starting from a 2-tilting torsion class  $\mathcal{E}$  in  $\mathcal{H}_{\mathcal{D}}$ , it is possible to define a new  $t$ -structure  $\mathcal{T}$  on  $D(\mathcal{H}_{\mathcal{D}})$  obtained by tilting the natural  $t$ -structure  $\mathcal{D}$  with respect to  $\mathcal{E}$ .

Hence, returning to 2-quasi-abelian categories, we will prove in Proposition 3.8 that we can associate to any 2-quasi-abelian category  $\mathcal{E}$  a canonical 2-tilting pair of  $t$ -structures  $(\mathcal{R}_{\mathcal{K}}, \mathcal{L}_{\mathcal{K}})$  on  $K(\mathcal{E})$  such that the intersection of the hearts is  $\mathcal{E}$ . This canonical choice corresponds to the “minimal” Quillen exact structure on  $\mathcal{E}$  and it produces as hearts the categories of coherent functors:  $\mathcal{RK}(\mathcal{E}) \cong (\mathcal{E}\text{-coh})^\circ$  and  $\mathcal{LK}(\mathcal{E}) \cong \text{coh-}\mathcal{E}$  (Corollary 3.9). We will prove that fixing any other Quillen exact structure  $(\mathcal{E}, \mathcal{E}x)$  it is possible to associate a canonical 2-tilting pair of  $t$ -structures  $(\mathcal{RD}_{\mathcal{E}x}, \mathcal{LD}_{\mathcal{E}x})$  on  $D(\mathcal{E}, \mathcal{E}x)$  such that  $\mathcal{E}$  is the intersection of the hearts (Lemma 3.10). Combining all these results we will prove the main Theorem 6.15 of this section which is the 2-version of Theorem 1.14.

**Lemma 3.1.** *Let  $(\mathcal{D}, \mathcal{T})$  be a 2-tilting pair of  $t$ -structures in  $\mathcal{C} \cong D(\mathcal{H}_{\mathcal{D}}) \cong D(\mathcal{H}_{\mathcal{T}})$ . Hence  $\mathcal{E} := \mathcal{H}_{\mathcal{D}} \cap \mathcal{H}_{\mathcal{T}}$  is closed under extensions (both in  $\mathcal{H}_{\mathcal{D}}$  and  $\mathcal{H}_{\mathcal{T}}$ ); it admits kernels and cokernels and given  $d : E \rightarrow F$  in  $\mathcal{E}$  we have  $\text{Ker}_{\mathcal{E}}(d) = \text{Ker}_{\mathcal{H}_{\mathcal{T}}}(d) \in \mathcal{E}$  while  $\text{Coker}_{\mathcal{E}}(d) = \text{Coker}_{\mathcal{H}_{\mathcal{D}}}(d) \in \mathcal{E}$ . Moreover the inclusion functor  $i : \mathcal{E} \rightarrow \mathcal{H}_{\mathcal{D}}$  admits a right adjoint  $t : \mathcal{H}_{\mathcal{D}} \rightarrow \mathcal{E}$  while the inclusion functor  $j : \mathcal{E} \rightarrow \mathcal{H}_{\mathcal{T}}$  admits a left adjoint  $f : \mathcal{H}_{\mathcal{T}} \rightarrow \mathcal{E}$ .*

*Proof.* Let  $d : E \rightarrow F$  be a morphism in  $\mathcal{E}$ , by point (2) of 2.1 we have:  $\text{Ker}_{\mathcal{H}_{\mathcal{T}}} d \in \mathcal{E}$  while  $\text{Coker}_{\mathcal{H}_{\mathcal{D}}} d \in \mathcal{E}$  and so they provide the kernel respectively the cokernel of  $d$  in  $\mathcal{E}$ . Let  $A \in \mathcal{H}_{\mathcal{D}}$  and let denote by  $\tau^{\leq 0}$  the truncation functor of the  $t$ -structure  $\mathcal{T}$ . The distinguished triangle  $\tau^{\leq 0}(A) \rightarrow A \rightarrow \tau^{\geq 1}(A) \xrightarrow{+}$  proves that  $\tau^{\leq 0}(A) \in \mathcal{D}^{\geq 0}$  since  $A \in \mathcal{H}_{\mathcal{D}}$  and  $\tau^{\geq 1}(A) \in \mathcal{T}^{\geq 1} \subseteq \mathcal{D}^{\geq -1}$  so  $t(A) := \tau^{\leq 0}(A) \in \mathcal{T}^{\leq 0} \cap \mathcal{D}^{\geq 0} = \mathcal{E}$ . Hence we have  $\mathcal{H}_{\mathcal{D}}(i(E), A) = \mathcal{C}(E, A) \cong \mathcal{C}(E, \tau^{\leq 0}A) = \mathcal{E}(E, t(A))$  for any  $E \in \mathcal{E}$  which proves that  $t$  is a right adjoint of  $i$ . Dually the functor  $\delta^{\geq 0}$  restricted to  $\mathcal{H}_{\mathcal{T}}$  takes image in  $\mathcal{E}$  and provides the left adjoint  $f$  of  $j$ .  $\square$

In the hierarchy of  $n$ -quasi-abelian categories the 2-level is defined in the following way:

**Definition 3.2.** <sup>1</sup> An additive category  $\mathcal{E}$  is said *2-quasi-abelian* if it admits kernels and cokernels. Hence any 1-quasi-abelian category is also 2-quasi-abelian.

At the 2-level a tilting torsion class is:

**Definition 3.3.** Let  $\mathcal{A}$  be an abelian category. A full subcategory  $\mathcal{E} \hookrightarrow \mathcal{A}$  is a *2-tilting torsion class* if

- (1)  $\mathcal{E}$  cogenerates  $\mathcal{A}$ ;
- (2)  $\mathcal{E}$  is closed under extensions in  $\mathcal{A}$ ;
- (3)  $\mathcal{E}$  has kernels;
- (4) for any exact sequence  $0 \rightarrow A \rightarrow X_1 \rightarrow X_2 \rightarrow B \rightarrow 0$  in  $\mathcal{A}$  with  $X \in \mathcal{E}$  and  $A, B \in \mathcal{A}$  we have  $B \in \mathcal{E}$ .

Hence any 1-tilting torsion class as in Definition 1.4 is also a 2-tilting torsion class. Dually a *2-cotilting torsion-free class* in  $\mathcal{A}$  is a full generating extension closed subcategory  $\mathcal{E}$  of  $\mathcal{A}$  admitting cokernels and closed under kernels in  $\mathcal{A}$ . Hence any 1-tilting torsion-free class as in Definition 1.4 (see also Lemma 1.5) is also a 2-tilting torsion-free class.

<sup>1</sup>This notion is sometimes called *pre-abelian* by some authors.

**Remark 3.4.** Any 2-tilting torsion class  $\mathcal{E}$  is a 2-quasi-abelian category since by Definition 3.3 (3) it admits kernels and by (4) it admits cokernels.

**Proposition 3.5.** *Given  $(\mathcal{D}, \mathcal{T})$  a 2-tilting pair of  $t$ -structures the category  $\mathcal{E} := \mathcal{H}_{\mathcal{D}} \cap \mathcal{H}_{\mathcal{T}}$  is a 2-tilting torsion class (respectively a 2-tilting torsion-free class) in  $\mathcal{H}_{\mathcal{D}}$  (respectively in  $\mathcal{H}_{\mathcal{T}}$ ).*

*Proof.* By Definition 2.4  $\mathcal{E}$  cogenerates  $\mathcal{H}_{\mathcal{D}}$  and generates  $\mathcal{H}_{\mathcal{T}}$ . By point (4) of 2.1 we have that  $\mathcal{E}$  is closed under extensions both in  $\mathcal{H}_{\mathcal{D}}$  and  $\mathcal{H}_{\mathcal{T}}$ . Given a morphism  $d : E_{-1} \rightarrow E_0$  in  $\mathcal{E}$  by point (2) of 2.1 we deduce that  $\text{Ker}_{\mathcal{E}} d \cong \text{Ker}_{\mathcal{H}_{\mathcal{T}}} d \in \mathcal{E}$  and  $\text{Coker}_{\mathcal{E}} d \cong \text{Coker}_{\mathcal{H}_{\mathcal{D}}} d \in \mathcal{E}$  which concludes the proof.  $\square$

**Theorem 3.6.** *Let  $\mathcal{D}$  be the natural  $t$ -structure on the triangulated category  $D(\mathcal{H}_{\mathcal{D}})$  and  $i : \mathcal{E} \rightarrow \mathcal{H}_{\mathcal{D}}$  a 2-tilting torsion class on  $\mathcal{H}_{\mathcal{D}}$ . Hence  $\mathcal{T}^{\leq 0} := \mathcal{D}^{\leq -2} \star \mathcal{E} \star \mathcal{E}[1]$  is an aisle in  $D(\mathcal{H}_{\mathcal{D}})$  (see C.2) such that  $\mathcal{E} = \mathcal{H}_{\mathcal{D}} \cap \mathcal{H}_{\mathcal{T}}$  and the pair  $(\mathcal{D}, \mathcal{T})$  is a 2-tilting pair of  $t$ -structures. We will say that the  $t$ -structure  $\mathcal{T}$  is obtained by tilting  $\mathcal{D}$  with respect to the 2-tilting torsion class  $\mathcal{E}$ .*

*Proof.* We note that  $\mathcal{T}^{\leq 0} := \mathcal{D}^{\leq -2} \star \mathcal{E} \star \mathcal{E}[1]$  is extension closed since any factor of this  $\star$  product is extension closed (see Appendix C.2). Moreover  $\mathcal{T}^{\leq 0}[1] \subseteq \mathcal{T}^{\leq 0}$  since the suspension of any factor is contained in a factor. By definition  $\mathcal{D}^{\leq -2} \subseteq \mathcal{T}^{\leq 0}$  and since any factor is contained in  $\mathcal{D}^{\leq 0}$  (which is extension closed) we have  $\mathcal{T}^{\leq 0} \subseteq \mathcal{D}^{\leq 0}$ . It remains to prove that the functor  $i_{\mathcal{T}^{\leq 0}} : \mathcal{T}^{\leq 0} \rightarrow D(\mathcal{H}_{\mathcal{D}})$  has a right adjoint  $\tau^{\leq 0} : D(\mathcal{H}_{\mathcal{D}}) \rightarrow \mathcal{T}^{\leq 0}$ . We will first define the restriction  $\tau_{[\mathcal{D}^{[-1,0]}]}^{\leq 0}$ , next we will prove how to extend the functor  $\tau_{[\mathcal{D}^{[-1,0]}]}^{\leq 0}$  to the whole  $D(\mathcal{H}_{\mathcal{D}})$ . Let us notice that the functor  $i : \mathcal{E} \rightarrow \mathcal{H}_{\mathcal{D}}$  has a right adjoint  $t$  defined as in Lemma 1.5: for any  $A \in \mathcal{H}_{\mathcal{D}}$  let consider a copresentation  $0 \rightarrow A \rightarrow X_1 \xrightarrow{f} X_2$  and let us pose  $\tau(A) = \text{Ker}_{\mathcal{E}}(f)$ . We note that if  $L \in \mathcal{D}^{\leq -2} \star \mathcal{E}$  hence  $H_{\mathcal{D}}^0(L) \in \mathcal{E}$ . For any  $M \in \mathcal{T}^{\leq 0}$  there exists a distinguished triangle  $L \rightarrow M \rightarrow X[1] \xrightarrow{+}$  with  $L \in \mathcal{D}^{\leq -2} \star \mathcal{E}$  and  $X \in \mathcal{E}$ . By applying to the previous triangle the homological functor associated to the  $t$ -structure  $\mathcal{D}$  we obtain  $X \rightarrow H_{\mathcal{D}}^0(L) \rightarrow H_{\mathcal{D}}^0(M) \rightarrow 0$  which proves that  $H_{\mathcal{D}}^0(M) \in \mathcal{E}$  since it is a cokernel in  $\mathcal{H}_{\mathcal{D}}$  between two objects in  $\mathcal{E}$ . Let  $A \in \mathcal{H}_{\mathcal{D}}$ , hence for any  $M \in \mathcal{T}^{\leq 0}$  we have  $\mathcal{C}(M, A) = \mathcal{H}_{\mathcal{D}}(H_{\mathcal{D}}^0(M), A) = \mathcal{H}_{\mathcal{D}}(H_{\mathcal{D}}^0(M), t(A)) \cong \mathcal{C}(M, t(A))$ . So our truncation functor  $\tau^{\leq 0}$  restricted to  $\mathcal{H}_{\mathcal{D}}$  coincides with  $t$ :  $\tau_{[\mathcal{H}_{\mathcal{D}}]}^{\leq 0} = t$ . Let us now compute the restriction of  $\tau^{\leq 0}$  to  $\mathcal{D}^{[-1,0]}$ . Given any object in  $D \in \mathcal{D}^{[-1,0]}$  there exists  $f : A \rightarrow B$  in  $\mathcal{H}_{\mathcal{D}}$  such that  $D$  is the mapping cone of the morphism  $f : A[0] \rightarrow B[0]$  in  $D(\mathcal{H}_{\mathcal{D}})$ . Since  $\mathcal{E}$  is cogenerating there exists an immersion  $h : A \hookrightarrow E$  with  $E \in \mathcal{E}$  and so  $D$  is isomorphic in  $D(\mathcal{H}_{\mathcal{D}})$  to the mapping cone of  $\bar{f} : E \rightarrow E \oplus_A B$ . The following commutative diagram (whose rows and columns are distinguished triangles) proves that  $\tau^{\leq 0}(D)$  is the mapping cone of  $E \rightarrow t(E \oplus_A B)$ :

$$\begin{array}{ccccccc}
E & \xrightarrow{t(\bar{f})} & t(E \oplus_A B) & \longrightarrow & \tau^{\leq 0}(D) & \xrightarrow{+} & \\
\downarrow & & \downarrow & & \downarrow & & \\
E & \xrightarrow{\bar{f}} & E \oplus_A B & \longrightarrow & D & \xrightarrow{+} & \\
\downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \tau^{\geq 1}(E \oplus_A B) & \longrightarrow & \tau^{\geq 1}(E \oplus_A B) & \xrightarrow{+} & \\
\downarrow^+ & & \downarrow^+ & & \downarrow^+ & & 
\end{array}$$

Now we are able to compute the truncation  $\tau^{\leq 0}(D)$  for any  $D \in \mathcal{D}^{[-1,0]}$ . Since  $\mathcal{T}^{\leq 0} \subseteq \mathcal{D}^{\leq 0}$  we have  $\tau^{\leq 0}(X) \cong \tau^{\leq 0}\delta^{\leq 0}(X)$  for any  $X \in \mathcal{C}$  (one can see by the octahedron axiom that the mapping cone of the composition  $\tau^{\leq 0}\delta^{\leq 0}(X) \rightarrow \delta^{\leq 0}(X) \rightarrow X$  lies in  $\mathcal{T}^{\geq 1}$ ). We can conclude since for any  $C \in \mathcal{D}^{\leq 0}$  the following commutative diagram (whose rows and columns are distinguished triangles):

$$\begin{array}{ccccccc}
\delta^{\leq -2}(C) & \longrightarrow & \tau^{\leq 0}(C) & \longrightarrow & \tau^{\leq 0}\delta^{[-1,0]}(C) & \xrightarrow{+} & \\
\downarrow & & \downarrow & & \downarrow & & \\
\delta^{\leq -2}(C) & \longrightarrow & C & \longrightarrow & \delta^{[-1,0]}(C) & \xrightarrow{+} & \\
\downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \tau^{\geq 1}\delta^{[-1,0]}(C) & \longrightarrow & \tau^{\geq 1}\delta^{[-1,0]}(C) & \xrightarrow{+} & \\
\downarrow^+ & & \downarrow^+ & & \downarrow^+ & & 
\end{array}$$

permits us to compute the  $\tau^{\leq 0}(C)$  for any  $C \in \mathcal{D}^{\leq 0}$ . The functoriality of this construction is guaranteed by the orthogonality of the classes  $\mathcal{T}^{\leq 0}, \mathcal{T}^{\geq 1}$ .

Let us prove that  $\mathcal{E} = \mathcal{H}_{\mathcal{D}} \cap \mathcal{H}_{\mathcal{T}}$ . Let consider  $A^\bullet \in \mathcal{H}_{\mathcal{D}} \cap \mathcal{H}_{\mathcal{T}}$ , hence  $A^\bullet \in \mathcal{T}^{\leq 0} = \mathcal{D}^{\leq -2} \star \mathcal{E} \star \mathcal{E}[1]$  and so it fits into a distinguished triangle  $B^\bullet \rightarrow A^\bullet \rightarrow E^{[-1,0]} \xrightarrow{+}$  for suitable  $B^\bullet \in \mathcal{D}^{\leq -2}$  and  $E^{[-1,0]} \in \mathcal{E} \star \mathcal{E}[1]$ ; but since  $A^\bullet \in \mathcal{D}^{\geq 0}$  we deduce that  $B^\bullet \in \mathcal{D}^{\leq -2} \cap \mathcal{D}^{\geq 0} = 0$  so  $A^\bullet \in \mathcal{E} \star \mathcal{E}[1]$ . Therefore  $A^\bullet = [E^{-1} \xrightarrow{d} \dot{E}^0]$  and  $A^\bullet \cong H_{\mathcal{D}}^0(A^\bullet)$  since  $A^\bullet \in \mathcal{H}_{\mathcal{D}}$ , so by point (4) of Definition 3.3 we obtain  $A^\bullet \in \mathcal{E}$  which proves that  $\mathcal{T}^{\leq 0} \cap \mathcal{D}^{\geq 0} = \mathcal{E}$ . We can apply the Tilting Theorem 2.3 ( $\mathcal{E}$  cogenerates  $\mathcal{H}_{\mathcal{D}}$ ) thus obtaining that  $(\mathcal{D}, \mathcal{T})$  is a 2-tilting pair of  $t$ -structures.  $\square$

**Remark 3.7.** The previous Theorem admits a dual version: given  $\mathcal{T}$  a  $t$ -structure on  $D(\mathcal{H}_{\mathcal{T}})$  and  $j : \mathcal{E} \rightarrow \mathcal{H}_{\mathcal{T}}$  a 2-cotilting torsion-free class on  $\mathcal{H}_{\mathcal{T}}$ , the class  $\mathcal{D}^{\geq 0} := \mathcal{E} \star \mathcal{E}[1] \star \mathcal{T}^{\geq 2}$  is a co-aisle in  $\mathcal{C}$  such that  $\mathcal{H}_{\mathcal{T}} \cap \mathcal{H}_{\mathcal{D}} = \mathcal{E}$ .

**Proposition 3.8.** *Let  $\mathcal{E}$  be a 2-quasi-abelian category. The category  $K(\mathcal{E})$  admits a canonical 2-tilting pair of  $t$ -structures  $(\mathcal{RK}_{\mathcal{E}}, \mathcal{LK}_{\mathcal{E}})$  such that  $\mathcal{E} = \mathcal{RK}(\mathcal{E}) \cap \mathcal{LK}(\mathcal{E})$  and so  $\mathcal{E} \hookrightarrow \mathcal{RK}(\mathcal{E})$  is a 2-tilting torsion class while  $\mathcal{E} \hookrightarrow \mathcal{LK}(\mathcal{E})$  is a 2-cotilting torsion-free class.*

*Proof.* In 1.12 we recall the construction of the left and right  $t$ -structures on  $K(\mathcal{E})$  with  $\mathcal{E}$  a quasi-abelian category. This construction is based on the existence of kernels and cokernels and so it can be adapted to any 2-quasi-abelian category thus providing the  $t$ -structures  $\mathcal{LK}_{\mathcal{E}} := (K^{\leq 0}(\mathcal{E}), (K^{\leq -1}(\mathcal{E}))^\perp)$  and  $\mathcal{RK}_{\mathcal{E}} = ({}^\perp(K^{\geq 1}(\mathcal{E})), K^{\geq 0}(\mathcal{E}))$  on  $K(\mathcal{E})$  whose associated truncated functors are those described in 1.12. Moreover  $\mathcal{RK}_{\mathcal{E}}^{\leq -2} \subseteq \mathcal{LK}_{\mathcal{E}}^{\leq 0} \subseteq \mathcal{RK}_{\mathcal{E}}^{\leq 0}$ . The heart of  $\mathcal{LK}_{\mathcal{E}}$  (respectively  $\mathcal{RK}_{\mathcal{E}}$ ) is denoted by  $\mathcal{LK}(\mathcal{E})$  (respectively  $\mathcal{RK}(\mathcal{E})$ ). Given a short exact sequence  $0 \rightarrow K \xrightarrow{\alpha} L \xrightarrow{\pi} E \rightarrow 0$  in  $\mathcal{LK}(\mathcal{E})$  with  $E \in \mathcal{E}$  it gives a distinguished triangle in  $K(\mathcal{E})$  and so  $\pi$  induces an isomorphism  $M(\alpha) \cong E$  in  $K(\mathcal{E})$  whose inverse provides a right inverse for  $\pi$  (which is therefore a split epimorphism). Hence  $\mathcal{E}$  coincides with the class of projective objects in  $\mathcal{LK}(\mathcal{E})$  and moreover any object  $L \in \mathcal{LK}(\mathcal{E})$  can be represented as a complex  $L \cong C(d) := [\text{Ker}_{\mathcal{E}}(d) \xrightarrow{\alpha} X \xrightarrow{d} \dot{Y}] \in K(\mathcal{E})$  (the other entries of the complex are 0 see Appendix C.3) since  $L \cong \tau_{\mathcal{E}}^{\geq 0} \tau_{\mathcal{E}}^{\leq 0} L$  (see 1.12). Thus  $L$  has a projective resolution of at most length 2 in the following way: the distinguished triangles (where  $C(\alpha) := [\text{Ker}_{\mathcal{E}}(d) \xrightarrow{\alpha} \dot{X}]$ )

$$(1) \quad \text{Ker}_{\mathcal{E}}(d)[0] \longrightarrow X[0] \longrightarrow C(\alpha) \xrightarrow{+} \quad C(\alpha) \longrightarrow Y[0] \longrightarrow C(d) \xrightarrow{+}$$

give the short exact sequences

$$0 \rightarrow \text{Ker}_{\mathcal{E}}(d) \rightarrow X \rightarrow C(\alpha) \rightarrow 0 \quad 0 \rightarrow C(\alpha) \rightarrow Y \rightarrow C(d) \rightarrow 0 ; \text{ in } \mathcal{K}(\mathcal{E})$$

from which we obtain the projective resolution  $0 \rightarrow \text{Ker}_{\mathcal{E}}(d) \rightarrow X \rightarrow Y \rightarrow C(d) \rightarrow 0$  of  $C(d)$  in  $\mathcal{K}(\mathcal{E})$ .

By Lemma C.10 we have  $K(\mathcal{E}) \cong D(\mathcal{L}\mathcal{K}(\mathcal{E}))$  which proves that  $(\mathcal{R}\mathcal{K}_{\mathcal{E}}, \mathcal{L}\mathcal{K}_{\mathcal{E}})$  is a 2-tilting pair of  $t$ -structures and hence by Proposition 3.5  $\mathcal{E}$  is a 2-tilting torsion (respectively 2-cotilting torsion-free) class in  $\mathcal{H}_{\mathcal{D}}$  (respectively  $\mathcal{H}_{\mathcal{T}}$ ).  $\square$

The previous proof suggests to interpret the hearts of the left and right  $t$ -structures in terms of coherent functors via the Yoneda Lemma (see Appendix B).

**Corollary 3.9.** *If  $\mathcal{E}$  is a 2-quasi-abelian category hence*

$$\mathcal{L}\mathcal{K}(\mathcal{E}) \cong \text{coh-}\mathcal{E} \quad \mathcal{R}\mathcal{K}(\mathcal{E}) \cong (\mathcal{E}\text{-coh})^{\circ}.$$

*Proof.* Since  $\mathcal{E}$  has kernels and cokernels it is a coherent category (see Definition B.11 and Proposition B.17). Both  $\text{coh-}\mathcal{E}$  and  $\mathcal{L}\mathcal{K}(\mathcal{E})$  are abelian categories whose projective objects coincide with  $\mathcal{E}$  and such that any object has a projective resolution of at most length 2. The functor  $I_{\mathcal{L}} : \mathcal{E} \rightarrow \mathcal{L}\mathcal{K}(\mathcal{E})$  extends uniquely to a functor  $I_{\mathcal{L}}^c : \text{coh-}\mathcal{E} \rightarrow \mathcal{L}\mathcal{K}(\mathcal{E})$  cokernel preserving (see B.10) which is an equivalence of categories (any object in  $L \in \mathcal{L}\mathcal{K}(\mathcal{E})$  has a projective resolution therefore  $I_{\mathcal{L}}^c$  is essentially surjective and fully faithful since any object in  $\mathcal{E}$  is projective in  $\mathcal{L}\mathcal{K}(\mathcal{E})$ ). Thus the left heart is equivalent to the category of right coherent functors. The right statement follows dually.  $\square$

**Lemma 3.10.** *Given any Quillen exact structure on  $(\mathcal{E}, \mathcal{E}x)$  the left and right  $t$ -structures induce a 2-tilting pair of  $t$ -structures  $(\mathcal{R}\mathcal{D}_{(\mathcal{E}, \mathcal{E}x)}, \mathcal{L}\mathcal{D}_{(\mathcal{E}, \mathcal{E}x)})$  on the derived category  $D(\mathcal{E}, \mathcal{E}x)$  such that  $\mathcal{E} = \mathcal{R}\mathcal{H}(\mathcal{E}, \mathcal{E}x) \cap \mathcal{L}\mathcal{H}(\mathcal{E}, \mathcal{E}x)$ .*

*Proof.* Let us prove that the  $t$ -structure  $\mathcal{L}\mathcal{K}_{\mathcal{E}}$  on  $K(\mathcal{E})$  satisfies the hypothesis of Proposition C.12 thus inducing (passing through the quotient) the  $t$ -structure  $\mathcal{L}\mathcal{D}_{(\mathcal{E}, \mathcal{E}x)}$  on  $D(\mathcal{E}, \mathcal{E}x) := K(\mathcal{E})/\mathcal{N}_{\mathcal{E}x}$ . We have to prove that, denoted by  $\mathcal{N}_{\mathcal{E}x}$  the null system of acyclic complexes with respect to  $(\mathcal{E}, \mathcal{E}x)$  (see Definition A.4), given any distinguished triangle  $Y^{\bullet} \rightarrow X^{\bullet} \rightarrow N^{\bullet} \xrightarrow{+}$  in  $K(\mathcal{E})$  such that  $Y^{\bullet} \in \mathcal{L}\mathcal{K}_{\mathcal{E}}^{\geq 1} = (K^{\leq 0}(\mathcal{E}))^{\perp}$ ,  $X^{\bullet} \in \mathcal{L}\mathcal{K}_{\mathcal{E}}^{\leq 0} = K^{\leq 0}(\mathcal{E})$  and  $N^{\bullet} \in \mathcal{N}_{\mathcal{E}x}$  we have  $Y^{\bullet}, X^{\bullet} \in \mathcal{N}_{\mathcal{E}x}$ . We can suppose  $Y^{\bullet} = \tau_{\mathcal{L}}^{\geq 1} Y^{\bullet}$  and  $X^{\bullet} \in K^{\leq 0}(\mathcal{E})$ . Hence the following commutative diagram permits to conclude that both  $X^{\bullet} \in Y^{\bullet} \in \mathcal{N}_{\mathcal{E}x}$ :

$$\begin{array}{ccccccc} \longrightarrow & 0 & \xrightarrow{\quad} & \text{Ker}(d_Y^0) & \xrightarrow{\quad} & Y^0 & \xrightarrow{d_Y^0} Y^1 \\ \downarrow & \downarrow & & \downarrow & & \downarrow & \downarrow \\ \longrightarrow & X^{-2} & \xrightarrow{d_X^{-2}} & X^{-1} & \xrightarrow{d_X^{-1}} & X^0 & \longrightarrow 0 \\ \downarrow & \downarrow & & \downarrow & & \downarrow & \downarrow \\ \longrightarrow & \text{Ker}(d_Y^0) \oplus X^{-2} & \xrightarrow{\quad} & Y^0 \oplus X^{-1} & \xrightarrow{\quad} & Y^1 \oplus X^0 & \longrightarrow Y^2 \\ \nwarrow \text{Im}(d_X^{-3}) & \nearrow & \nwarrow \text{Ker}(d_Y^0) \oplus \text{Im}(d_X^{-2}) & \nearrow & \nwarrow \text{Ker}(d_Y^1) \oplus X^0 & \nearrow & \nwarrow \text{Ker}(d_Y^2) \end{array}$$

(one has to start looking the last row, for  $i \leq -3$  we have  $N^i = X^i$  while for  $j \geq 1$  we have  $N^j = Y^{j+1}$  so we can write  $\text{Im}(d_X^{-3})$  on the left and  $\text{Ker}(d_Y^2)$  on the right, hence we complete taking respectively the cokernel and the kernel and we are able to decompose  $Y^{\bullet}$  and  $X^{\bullet}$  via conflations). Therefore we obtain a pair of  $t$ -structures  $(\mathcal{R}\mathcal{D}_{(\mathcal{E}, \mathcal{E}x)}, \mathcal{L}\mathcal{D}_{(\mathcal{E}, \mathcal{E}x)})$  on  $D(\mathcal{E}, \mathcal{E}x)$  such that  $\mathcal{R}\mathcal{D}_{(\mathcal{E}, \mathcal{E}x)}^{\leq -2} \subseteq \mathcal{L}\mathcal{D}_{(\mathcal{E}, \mathcal{E}x)}^{\leq 0} \subseteq \mathcal{R}\mathcal{D}_{(\mathcal{E}, \mathcal{E}x)}^{\leq 0}$ .

Clearly  $\mathcal{E} \subseteq \mathcal{R}\mathcal{H}(\mathcal{E}, \mathcal{E}x) \cap \mathcal{L}\mathcal{H}(\mathcal{E}, \mathcal{E}x)$ . If  $E^{\bullet} \in \mathcal{R}\mathcal{H}(\mathcal{E}, \mathcal{E}x) \cap \mathcal{L}\mathcal{H}(\mathcal{E}, \mathcal{E}x)$ , we can suppose  $E^{\bullet} \in K^{\leq 0}(\mathcal{E})$  and, since it belongs to  $\mathcal{R}\mathcal{H}(\mathcal{E}, \mathcal{E}x)$ , we have that  $E^{\bullet} \rightarrow$

$\tau_{\mathcal{R}}^{\geq 0} E^\bullet = \text{Coker}_{\mathcal{E}} d_{E^\bullet}^{-1}[0] \in \mathcal{E}$  is a quasi-isomorphism (i.e. its mapping cone belongs to  $\mathcal{N}_{\mathcal{E}x}$ ) and so  $E^\bullet \in \mathcal{E}$ .

It remains to prove that the derived category of the heart is equivalent to  $D(\mathcal{LH}(\mathcal{E}, \mathcal{E}x)) \cong D(\mathcal{E}, \mathcal{E}x) =: K(\mathcal{E})/\mathcal{N}_{\mathcal{E}x}$ . Now  $\mathcal{E}$  is a full subcategory of  $\mathcal{LH}(\mathcal{E}, \mathcal{E}x)$  and a sequence  $S : 0 \rightarrow E_1 \rightarrow E \rightarrow E_2 \rightarrow 0$  is exact for the Quillen exact structure  $(\mathcal{E}, \mathcal{E}x)$  if and only if the triangle  $E_1[0] \rightarrow E[0] \rightarrow E_2[0] \xrightarrow{+}$  is distinguished in  $D(\mathcal{E}, \mathcal{E}x)$  and hence (since any term is in  $\mathcal{LH}(\mathcal{E}, \mathcal{E}x)$ ) if and only if  $S$  is exact in  $\mathcal{LH}(\mathcal{E}, \mathcal{E}x)$ . We note that given any morphism  $f : E \rightarrow F$  in  $\mathcal{E}$  we have  $\text{Ker}_{\mathcal{LH}(\mathcal{E}, \mathcal{E}x)}(f) = H_{\mathcal{LH}(\mathcal{E}, \mathcal{E}x)}^0([E \rightarrow F]) \in \mathcal{E}$  (due to the inclusion  $\mathcal{RD}_{(\mathcal{E}, \mathcal{E}x)}^{\leq -2} \subseteq \mathcal{LD}_{(\mathcal{E}, \mathcal{E}x)}^{\leq 0}$ ) and so  $\text{Ker}_{\mathcal{LH}(\mathcal{E}, \mathcal{E}x)}(f) = \text{Ker}_{\mathcal{E}}(f)$ . Hence any complex in  $K(\mathcal{E})$  which is acyclic in  $D(\mathcal{LH}(\mathcal{E}, \mathcal{E}x))$  can be decomposed into short exact sequences in  $\mathcal{LH}(\mathcal{E}, \mathcal{E}x)$  whose terms belong to  $\mathcal{E}$  and so we deduce that  $\mathcal{N}_{\mathcal{E}x} = \mathcal{N}_{\mathcal{LH}(\mathcal{E}, \mathcal{E}x)} \cap K(\mathcal{E})$ . Moreover any object  $X^\bullet \in \mathcal{LH}(\mathcal{E}, \mathcal{E}x)$  can be represented as a complex  $X^\bullet \in K^{\leq 0}(\mathcal{E})$  such that  $\tau_{\mathcal{E}}^{\geq 0} X^\bullet \cong X^\bullet$  and so (as in the proof of Proposition 3.8) it can be represented by a complex  $C(d) := [\text{Ker}_{\mathcal{E}}(d) \xrightarrow{\alpha} X \xrightarrow{d} Y] \in \mathcal{LH}(\mathcal{E}, \mathcal{E}x)$  whose terms belong to  $\mathcal{E}$ . This suggest that  $\mathcal{LH}(\mathcal{E}, \mathcal{E}x)$  is a Gabriel quotient of the heart  $\mathcal{LK}(\mathcal{E})$  as we will see in Theorem 6.11. The same argument of Proposition 3.8 (1) proves that the exact sequence  $0 \rightarrow \text{Ker}_{\mathcal{E}}(d) \rightarrow X \rightarrow Y \rightarrow C(d) \rightarrow 0$  is exact in  $\mathcal{LH}(\mathcal{E}, \mathcal{E}x)$  and hence any object in the left heart admits a  $\mathcal{E}$ -resolution of length at most 2. Therefore the subcategory  $\mathcal{E}$  in  $\mathcal{LH}(\mathcal{E}, \mathcal{E}x)$  satisfies the hypotheses of [KS06, Proposition 13.2.6] (see Proposition C.11) and hence  $\frac{K(\mathcal{E})}{\mathcal{N}_{\mathcal{E}x}} \cong D(\mathcal{LH}(\mathcal{E}, \mathcal{E}x))$ .  $\square$

Now we have a candidate for any vertex of Theorem 1.14 square. The main difference with the 1-case is that we need to introduce the use of Quillen exact structures. We will denote by  $\{2\text{-quasi-abelian categories} + \mathcal{E}x\}$  the class whose objects are  $(\mathcal{E}, \mathcal{E}x)$  with  $\mathcal{E}$  a 2-quasi-abelian category and  $\mathcal{E}x$  a Quillen exact structure on  $\mathcal{E}$ .

**Theorem 3.11.** *There is a one-to-one correspondence between the classes*

$$\begin{array}{ccc}
\{2\text{-quasi-abelian categories} + \mathcal{E}x\} & \longleftrightarrow & \{2\text{-tilting pairs of } t\text{-structures}\} \\
\uparrow & & \uparrow \\
\mathcal{E} = \mathcal{RH}(\mathcal{E}, \mathcal{E}x) \cap \mathcal{LH}(\mathcal{E}, \mathcal{E}x) & \longleftrightarrow & (\mathcal{RD}_{(\mathcal{E}, \mathcal{E}x)}, \mathcal{LD}_{(\mathcal{E}, \mathcal{E}x)}) \text{ on } \mathcal{C} = D(\mathcal{E}, \mathcal{E}x) \\
\downarrow & & \downarrow \\
\{2\text{-tilting torsion classes}\} & \longleftrightarrow & \{2\text{-cotilting torsion-free classes}\} \\
\downarrow & & \downarrow \\
\mathcal{E} \text{ in } \mathcal{RH}(\mathcal{E}, \mathcal{E}x) & \longleftrightarrow & \mathcal{E} \text{ in } \mathcal{LH}(\mathcal{E}, \mathcal{E}x).
\end{array}$$

*Proof.* By Proposition 3.5 given any 2-tilting pair of  $t$ -structures  $(\mathcal{D}, \mathcal{T})$  we obtain that  $\mathcal{E}$  is a 2-tilting torsion (respectively 2-cotilting torsion-free) class in  $\mathcal{H}_{\mathcal{D}}$  (respectively  $\mathcal{H}_{\mathcal{T}}$ ) and by Remark 3.4  $\mathcal{E}$  is 2-quasi-abelian. By Theorem 3.6 given  $\mathcal{E}$  a 2-tilting torsion class in  $\mathcal{H}_{\mathcal{D}}$  the pair  $(\mathcal{D}, \mathcal{T})$  (on  $D(\mathcal{H}_{\mathcal{D}})$ ) is a 2-tilting pair of  $t$ -structures (where  $\mathcal{T}$  is the  $t$ -structure obtained by tilting  $\mathcal{D}$  with respect to  $\mathcal{E}$ ) and by Proposition 2.5 the pair  $(\mathcal{D}, \mathcal{T})$  coincides with  $(\mathcal{RD}_{(\mathcal{E}, \mathcal{E}x)}, \mathcal{LD}_{(\mathcal{E}, \mathcal{E}x)})$ . Given  $(\mathcal{E}, \mathcal{E}x)$  a 2-quasi-abelian category endowed with a Quillen exact structure by Lemma 3.10 one can associate the 2-tilting pair of  $t$ -structures  $(\mathcal{RD}_{(\mathcal{E}, \mathcal{E}x)}, \mathcal{LD}_{(\mathcal{E}, \mathcal{E}x)})$  on  $D(\mathcal{E}, \mathcal{E}x)$  such that  $\mathcal{E} = \mathcal{RH}(\mathcal{E}, \mathcal{E}x) \cap \mathcal{LH}(\mathcal{E}, \mathcal{E}x)$ .  $\square$



**Example 3.12.** Let  $R$  be a (left and right) coherent ring with global dimension  $\text{gl.dim}(R) = 1$  and  $\mathcal{E} := \text{add}(R)$  (see Appendix B.4). The maximal Quillen exact structure on  $\mathcal{E}$  coincides with the minimal one and  $\mathcal{E}$  is a 1-quasi-abelian category; its left heart is  $\mathcal{LK}(\mathcal{E}) \cong \text{coh-}R$  (and so  $\mathcal{E} = \text{proj-}\mathcal{E}$  is 1-cotilting torsion-free class with its minimal Quillen exact structure) while  $\mathcal{RK}(\mathcal{E}) \cong (\mathcal{E}\text{-coh})^\circ$ .

Let  $R$  be a (left and right) coherent ring with global dimension  $\text{gl.dim}(R) = 2$  and  $\mathcal{E} := \text{add}(R)$ . Hence  $\mathcal{E}$  is a 2-quasi-abelian category: for any  $f : P_1 \rightarrow P_2$  morphism in  $\mathcal{E}$  its kernel  $\text{Ker}_{\mathcal{E}}(f) = \text{Ker}_{\text{coh-}R}(f) \in \mathcal{E}$  due to the fact that the right projective dimension is at most 2 and the ring is coherent; while  $\text{Coker}_{\mathcal{E}}(f) = (\text{Ker}_{R\text{-coh}}(f^*))^*$  where  $(-)^* := \text{Hom}_R(-, R)$ . In [Rum08] constructed a tilted algebra  $A$  of type  $\mathbb{E}_6$  such that its category of projective modules of finite type is 2-quasi-abelian (since  $A$  has global dimension 2) but not 1-quasi-abelian.

So  $\mathcal{E}$  is a 2-quasi-abelian category but in certain cases it could be also 1-quasi-abelian. For example let us consider the affine plane  $\mathbb{A}_k^2 = \text{Spec}(R)$  with  $R = k[x, y]$  and  $k$  a field; hence  $R$  has projective dimension 2 and it is Noetherian and so coherent; this assures that  $\mathcal{E} := \text{add}(R)$  is a 2-quasi-abelian category. In this case  $\mathcal{E}$  coincides with the category of free  $R$ -modules of finite type (this result was proved by Seshadri in [Ses58] while the general statement known as Serre conjecture was proved by Quillen and Suslin [Qui76], [Sus76]) and its left heart as a 2-quasi-abelian category endowed with its minimal Quillen exact structure, is the category  $\text{coh-}R$  or equivalently the category  $\text{Coh}(\mathcal{O}_{\mathbb{A}_k^2})$  of coherent sheaves on the affine plane  $\mathbb{A}_k^2$ . A

sequence  $0 \rightarrow \mathcal{E}_1 \xrightarrow{\alpha} \mathcal{E}_2 \xrightarrow{\beta} \mathcal{E}_3 \rightarrow 0$  is exact in  $\mathcal{E}$  if and only if  $\mathcal{E}_3 \cong (\text{Ker}_R(\beta^*))^*$  and so the cokernel of  $\beta$  in  $\text{Coh}(\mathcal{O}_{\mathbb{A}_k^2})$  is a torsion sheaf whose support has dimension 0 (finite union of closed points). On the other side any coherent sheaf supported on a finite union of closed points can be represented as a cokernel of such a  $\beta$ . Let denote by  $\mathcal{T}_0$  the class of torsion sheaves supported on points; this is a Serre subcategory of  $\text{Coh}(\mathcal{O}_{\mathbb{A}_k^2})$  and the functor  $I_{\mathcal{L}} : \mathcal{E} \rightarrow \text{Coh}(\mathcal{O}_{\mathbb{A}_k^2})/\mathcal{T}_0$  is fully faithful and  $\mathcal{E}$  is a 1-cotilting torsion-free class in  $\text{Coh}(\mathcal{O}_{\mathbb{A}_k^2})/\mathcal{T}_0$  and so  $\mathcal{E}$  is 1-quasi-abelian category (an hence the left heart of  $\mathcal{E}$  as a 1-quasi-abelian category is the quotient abelian category  $\text{Coh}(\mathcal{O}_{\mathbb{A}_k^2})/\mathcal{T}_0$ ).

**Example 3.13.** Let  $\mathcal{E}$  be the category of free abelian groups of finite type. It is a quasi-abelian category and its maximal Quillen exact structure coincides with the minimal one (split short exact sequences). Its left heart  $\mathcal{LK}(\mathcal{E})$  is the whole category of finitely generated abelian groups while  $\mathcal{RK}(\mathcal{E}) = (\mathcal{E}\text{-coh})^\circ$  is equivalent to the opposite category of the category of abelian groups of finite type. The derived equivalence  $D(\mathcal{Ab}) \cong D(\mathcal{Ab}^\circ)$  is given by  $\mathbf{R}\text{Hom}_{\mathbb{Z}}(-, \mathbb{Z})$  and the intersection of the hearts is given by the finitely generated abelian groups  $F$  such that  $\mathbf{R}\text{Hom}_{\mathbb{Z}}(F, \mathbb{Z}) = \text{Hom}_{\mathbb{Z}}(F, \mathbb{Z})$  which are the free abelian groups of finite type. One can also interpret the right heart as the Happel Reiten Smalø right tilt of the abelian category of finitely generated abelian groups with respect to the cotilting torsion-free class of free abelian groups of finite type: i.e.; objects are complexes  $d : F_0 \rightarrow F_1$  (in degree 0 e 1) of free abelian groups such that  $\text{Coker}(d)$  is a torsion group.

**Example 3.14.** [Bay, Example 3.6.(5), Exercise 3.7.(12)]. Let  $X$  be a smooth projective curve,  $\mu \in \mathbb{R}$  a real number and let  $A_{\geq \mu}$  be the full subcategory of  $\text{Coh}(\mathcal{O}_X)$  generated by torsion sheaves and vector bundles whose HN-filtration quotients have slope  $\geq \mu$ . Hence  $A_{\geq \mu}$  is a tilting torsion class in  $\text{Coh}(\mathcal{O}_X)$ . In particular let  $X = \mathbb{P}_k^1$  the projective line over a field  $k$ . Let us recall that any coherent sheaf  $\mathcal{F} \in \text{Coh}(\mathcal{O}_{\mathbb{P}_k^1})$



decomposes as  $\mathcal{F} \cong \mathcal{F}_{\text{tor}} \oplus \mathcal{F}_{\text{free}}$  and, by the Birkhoff-Grothendieck theorem, the torsion-free part is a direct sum of line bundles  $\mathcal{O}_{\mathbb{P}_k^1}(d_i)$ . So  $\mathcal{E} := \mathcal{A}_{\geq 0}$  is a tilting torsion class in  $\text{Coh}(\mathcal{O}_{\mathbb{P}_k^1})$  (and hence it is a 1-quasi-abelian category). In this case the maximal Quillen exact structure on  $\mathcal{E}$  does not coincide with the minimal one since the sequence  $0 \rightarrow \mathcal{O}_{\mathbb{P}_k^1} \rightarrow \mathcal{O}_{\mathbb{P}_k^1}(1)^2 \rightarrow \mathcal{O}_{\mathbb{P}_k^1}(2) \rightarrow 0$  does not split (i.e.  $\text{Ext}_{\mathcal{O}_{\mathbb{P}_k^1}}^1(\mathcal{O}_{\mathbb{P}_k^1}(2), \mathcal{O}_{\mathbb{P}_k^1}) \neq 0$ ). So we have a right heart (as a 2-quasi-abelian category with  $\mathcal{E}$  endowed with the split exact structure) in  $K(\mathcal{E})$  which is the category  $(\mathcal{E}\text{-coh})^\circ$  while its right heart in  $D(\mathcal{E})$  as 1-quasi-abelian category is the category of coherent sheaves  $\text{Coh}(\mathcal{O}_{\mathbb{P}_k^1})$  (since  $\mathcal{E}$  is a 1-tilting torsion class in it). Concerning the left heart  $\mathcal{LD}(\mathcal{E})$  its objects are complexes  $X = [\mathcal{E}^{-1} \xrightarrow{d} \mathcal{E}^0]$  with  $\mathcal{E}^i \in \mathcal{E}$  and  $d$  a monomorphism in  $\mathcal{E}$ . Since any object in  $\mathcal{E}$  admits a finite resolution whose terms are direct factors of finite direct sums of  $\mathcal{O}_{\mathbb{P}_k^1} \oplus \mathcal{O}_{\mathbb{P}_k^1}(1)$  (and so in  $\text{add}(\mathcal{O}_{\mathbb{P}_k^1} \oplus \mathcal{O}_{\mathbb{P}_k^1}(1))$  see Appendix B.4) we can represent  $X$  as a bounded complex  $X = [X^{-m} \rightarrow \dots \rightarrow X^0] \in K^{\leq 0}(\text{add}(\mathcal{O}_{\mathbb{P}_k^1} \oplus \mathcal{O}_{\mathbb{P}_k^1}(1)))$ . Thus for any  $X \in \mathcal{LD}(\mathcal{E})$  and for any  $i > 0$  we have  $\text{Ext}^i(\mathcal{O}_{\mathbb{P}_k^1} \oplus \mathcal{O}_{\mathbb{P}_k^1}(1), X) \cong \mathcal{D}(\mathcal{E})(\mathcal{O}_{\mathbb{P}_k^1} \oplus \mathcal{O}_{\mathbb{P}_k^1}(1), X[i]) = 0$  and (via the associated distinguished triangle) we get a short exact sequence in the left heart  $0 \rightarrow X^{[-m, -1]}[-1] \rightarrow X^0 \rightarrow X \rightarrow 0$  which proves that  $T = \mathcal{O}_{\mathbb{P}_k^1} \oplus \mathcal{O}_{\mathbb{P}_k^1}(1)$  is a projective generator of the left heart  $\mathcal{LD}(\mathcal{E})$ . Hence  $\mathcal{LD}(\mathcal{E})$  is equivalent to the category of left modules of finite type on the ring  $R := \text{End}(\mathcal{O}_{\mathbb{P}_k^1} \oplus \mathcal{O}_{\mathbb{P}_k^1}(1))$  which is the path algebra of the Kronecker quiver  $Q$

$$\bullet \rightrightarrows \bullet$$

The derived equivalence  $D^b(\text{Coh}(\mathcal{O}_{\mathbb{P}_k^1})) \cong D^b(\text{Rep}_k(Q))$  (which holds true also in the unbounded derived categories) is due to A. Beilinson and  $T = \mathcal{O}_{\mathbb{P}_k^1} \oplus \mathcal{O}_{\mathbb{P}_k^1}(1)$  is an example of *tilting sheaf*.

**Remark 3.15.** We have proved that for any  $n$ -quasi-abelian category  $(\mathcal{E}, \mathcal{E}x)$  (with  $n \in \{1, 2\}$ ) we have a derived equivalence  $D(\mathcal{LD}(\mathcal{E}, \mathcal{E}x)) \cong D(\mathcal{RD}(\mathcal{E}, \mathcal{E}x))$  even if the category  $\mathcal{E}$  does not contain a tilting object.

#### 4. EFFACEABLE FUNCTORS

This section is devoted to the tool of effaceable functors which we will use in Section 6.

**Proposition 4.1.** *Let  $\mathcal{E}$  be a projectively complete category and let  $\text{fp-}\mathcal{E}$  be the Freyd category of (contravariant) finitely presented functors. The maximal Quillen exact structure on  $\text{fp-}\mathcal{E}$  is the one whose conflations are  $0 \rightarrow F_1 \rightarrowtail F \twoheadrightarrow F_2 \rightarrow 0$  such that for any  $E \in \mathcal{E}$  the sequences of abelian groups  $0 \rightarrow F_1(E) \rightarrow F(E) \rightarrow F_2(E) \rightarrow 0$  are exact.*

*Proof.* Let us recall that  $\text{fp-}\mathcal{E}$  admits cokernels which are calculated pointwise and if a morphism admits a kernel it is also computed pointwise; moreover by Proposition B.13 any functor which is (pointwise or in  $\text{Mod-}\mathcal{E}$ ) extension of finitely presented functors is finitely presented too. Hence the push-out of any inflation is an inflation, respectively the pull-back of any deflation is a deflation and they are stable by compositions so these conflations define a Quillen exact structure on  $\text{fp-}\mathcal{E}$ . For any other Quillen exact structure on  $\text{fp-}\mathcal{E}$  a conflation  $0 \rightarrow G_1 \rightarrowtail G \twoheadrightarrow G_2 \rightarrow 0$  is

a kernel-cokernel sequence and so for any  $E \in \mathcal{E}$  we get a short exact sequence  $0 \rightarrow G_1(E) \rightarrow G(E) \rightarrow G_2(E) \rightarrow 0$  of abelian groups.  $\square$

**Definition 4.2.** [Sch04, Definition 1.3.] Let  $\mathcal{U}$  be an exact category (i.e. an additive category with a Quillen exact structure) and  $\mathcal{A} \subset \mathcal{U}$ . Then the inclusion  $\mathcal{A} \subset \mathcal{U}$  is called *right filtering* and  $\mathcal{A}$  is called *right filtering in  $\mathcal{U}$*  if:

- (1)  $\mathcal{A}$  is an extension closed full subcategory of  $\mathcal{U}$ ;
- (2)  $\mathcal{A}$  is closed under taking admissible subobjects and admissible quotients in  $\mathcal{U}$ ;
- (3) every map  $f : U \rightarrow A$  with  $U \in \mathcal{U}$  and  $A \in \mathcal{A}$  admits a factorisation  $f = g\pi$   $U \xrightarrow{\pi} B \xrightarrow{g} A$  with  $B \in \mathcal{A}$  and  $\pi$  a deflation.

**Definition 4.3.** [Sch04, Definition 1.12.] Let  $\mathcal{U}$  be an exact category and  $\mathcal{A} \subset \mathcal{U}$  be an extension closed full subcategory. A  $\mathcal{U}$ -morphism is called a *weak isomorphism* if it is a finite composition of inflations with cokernel in  $\mathcal{A}$  and deflations with kernel in  $\mathcal{A}$ . We write  $\Sigma_{\mathcal{A} \subset \mathcal{U}}$  for the class of weak isomorphisms.

**Lemma 4.4.** [Sch04, Lemma 1.13.] *If  $\mathcal{A}$  is right filtering in  $\mathcal{U}$  then  $\Sigma_{\mathcal{A} \subset \mathcal{U}}$  admits a calculus of right fractions.*

By passing to the opposite category one obtains the dual results in the left filtering case.

In the following we will define a right filtering subcategory  $\text{eff-}\mathcal{E}_x$  of  $\text{fp-}\mathcal{E}$  whose objects are the quotients in  $\text{fp-}\mathcal{E}$  of deflations in  $\mathcal{E}_x$  and they are called effaceable functors following the classical definition see [Swa68, p.14], [Wei94, p.28] and [Kra15, p.4]. When  $\mathcal{A}$  is an abelian category the right orthogonal class of  $\text{eff-}\mathcal{A}$  coincides with the full subcategory of coherent functors which respects the monomorphisms and hence the quotient category  $\frac{\text{coh-}\mathcal{A}}{\text{eff-}\mathcal{A}}$  provides the category of coherent left exact functors thus obtaining the so called Auslander Formula:  $\mathcal{A} \cong \frac{\text{coh-}\mathcal{A}}{\text{eff-}\mathcal{A}}$  ([Kra15, Theorem 2.2]).

This procedure is analog to the procedure one needs to do in order to define the category of sheaves in abelian groups associated to a topological space. One first defines the localizing Serre subcategory of pre-sheaves which have stalk 0 at any point and hence its right orthogonal class is formed by separated pre-sheaves while the quotient category provides the category of sheaves in abelian groups.

It turns out that the approach via Quillen exact structures is equivalent to the one via Grothendieck topologies as recently explained by Kaledin and Lowen in their paper [KL15, 2.2, 2.5]. The deflations (respectively the inflations) of a Quillen exact structure provide a Grothendieck pre-topology in  $\mathcal{E}$  (respectively in  $\mathcal{E}^\circ$ ). In this equivalence the notion of pre-sheaf with stalk 0 at any point would give rise to the notion of weak effaceable functor which is equivalent to the notion of effaceable functor in the finitely presented case (see Proposition 4.5).

Following the analogy with abelian sheaves on a topological space  $X$ , a pre-sheaf  $\mathcal{F}$  has stalk 0 in any point  $x \in X$  if and only if for any  $U$  open subset of  $X$  and  $\eta \in \mathcal{F}(U)$  there exists an open covering  $p : \bigsqcup_{i \in I} U_i \rightarrow U$  such that the restriction  $\mathcal{F}(p)(\eta) = \prod_{i \in I} \eta|_{U_i} = 0$ . In the additive context we have the following counterpart:

**Proposition 4.5.** *Let  $\mathcal{E}$  be a projectively complete category endowed with a Quillen exact structure  $(\mathcal{E}, \mathcal{E}_x)$  and  $\text{fp-}\mathcal{E}$  its Freyd category; we denote by*

$$\text{eff-}\mathcal{E}_x := \{\text{Coker}_{\text{fp-}\mathcal{E}}(q) \mid q \text{ is a deflation in } \mathcal{E}_x\}$$

the full subcategory of  $\text{fp-}\mathcal{E}$  whose objects are cokernels of morphisms induced by deflations of  $\mathcal{E}$ . We call the elements of  $\text{eff-}_{\mathcal{E}x}\mathcal{E}$  effaceable functors.

The following are equivalent:

- (1)  $F \in \text{fp-}\mathcal{E}$  is effaceable;
- (2) for any  $U \in \mathcal{E}$  and  $\eta \in F(U)$  there exists a deflation  $p : Y \twoheadrightarrow U$  such that  $F(p)(\eta) = 0$  (weak effaceable).

*Proof.* Let us prove that (1)  $\Rightarrow$  (2). Let consider  $\mathcal{E}_{E_1} \xrightarrow{q} \mathcal{E}_{E_2} \rightarrow F \rightarrow 0$  with  $q : E_1 \twoheadrightarrow E_2$  a deflation in  $\mathcal{E}$ . We have to prove that for any  $\eta \in F(U) \cong \text{Hom}_{\text{fp-}\mathcal{E}}(\mathcal{E}_U, F)$  there exists a deflation  $p : Y \twoheadrightarrow U$  such that  $F(p)(\eta) = 0$ . Let consider the following commutative diagram where  $h$  exists since  $\mathcal{E}_U$  is projective in  $\text{fp-}\mathcal{E}$ ,  $Y := E_1 \times_{E_2} U$  and  $p$  is a deflation since it is the pull-back of a deflation:

$$\begin{array}{ccccc}
 & \mathcal{E}_Y & \xrightarrow{p} & \mathcal{E}_U & \\
 & \swarrow & & \downarrow \eta & \\
 \mathcal{E}_{E_1} & \xrightarrow{q} & \mathcal{E}_{E_2} & \xrightarrow{h} & F \longrightarrow 0. \\
 & \searrow & & & \\
 & 0 & & & 
 \end{array}$$

hence  $F(p)(\eta) = \eta p = 0$ .

Let us prove that (2)  $\Rightarrow$  (1). Since  $F \in \text{fp-}\mathcal{E}$  is finitely presented there exists  $f \in \mathcal{E}(E_1, E_2)$  such that  $\mathcal{E}_{E_1} \xrightarrow{f} \mathcal{E}_{E_2} \xrightarrow{\eta} F \rightarrow 0$  and by hypothesis (2) there exists a deflation  $p : Y \twoheadrightarrow E_2$  such that  $\eta p = 0$  hence (since  $\mathcal{E}_{E_1} \twoheadrightarrow \text{Ker}_{\text{fg-}\mathcal{E}}(\eta)$  and  $\mathcal{E}_Y$  is projective in  $\text{fg-}\mathcal{E}$ ) there exists  $g : Y \rightarrow E_1$  such that  $p = fg$  which proves (by Remark A.3) that  $f$  is a deflation too.  $\square$

**Remark 4.6.** Following (2) implies (1) in the previous Proposition 4.5 we have also proved that, given  $\mathcal{E}_{E_1} \xrightarrow{f} \mathcal{E}_{E_2} \xrightarrow{\eta} F \rightarrow 0$  any presentation of an effaceable functor, the map  $f$  is a deflation.

**Proposition 4.7.** Let consider  $\text{fp-}\mathcal{E}$  endowed with its maximal Quillen exact structure. The full subcategory  $\text{eff-}_{\mathcal{E}x}\mathcal{E} \subset \text{fp-}\mathcal{E}$  is right filtering; if  $\mathcal{E}$  is right coherent, hence  $\text{eff-}_{\mathcal{E}x}\mathcal{E}$  is a Serre subcategory of the abelian category  $\text{fp-}\mathcal{E} = \text{coh-}\mathcal{E}$ . Dually  $\mathcal{E}\text{-eff}_{\mathcal{E}x} \subset \mathcal{E}\text{-fp}$  is left filtering in  $\mathcal{E}\text{-fp}$  and if  $\mathcal{E}$  is left coherent, hence  $\mathcal{E}\text{-eff}_{\mathcal{E}x}$  is a Serre subcategory of the abelian category  $\mathcal{E}\text{-fp} = \mathcal{E}\text{-coh}$ .

*Proof.* Let us prove that  $\text{eff-}_{\mathcal{E}x}\mathcal{E} \subset \text{fp-}\mathcal{E}$  is right filtering; by Definition 4.2 we have to verify:

- (1)  $\text{eff-}_{\mathcal{E}x}\mathcal{E}$  is an extension closed full subcategory of  $\text{fp-}\mathcal{E}$ ;
- (2)  $\text{eff-}_{\mathcal{E}x}\mathcal{E}$  is closed under taking admissible subobjects and admissible quotients in  $\text{fp-}\mathcal{E}$ ;
- (3) every map  $f : U \rightarrow A$  with  $U \in \text{fp-}\mathcal{E}$  and  $A \in \text{eff-}_{\mathcal{E}x}\mathcal{E}$  admits a factorisation  $f = g\pi$  with  $U \xrightarrow{\pi} B \xrightarrow{g} A$ ,  $\pi$  a deflation and  $B \in \text{eff-}_{\mathcal{E}x}\mathcal{E}$ .

Let us verify that  $\text{eff-}_{\mathcal{E}x}\mathcal{E}$  is closed under extension in  $\text{fp-}\mathcal{E}$ . Let consider a conflation  $0 \rightarrow T_1 \rightarrow T \rightarrow T_2 \rightarrow 0$  such that both  $T_1, T_2$  are effaceable functors and  $\eta \in T(U) \cong \text{fp-}\mathcal{E}(\mathcal{E}_U, T)$  with  $U \in \mathcal{E}$ . Let us look at the following commutative diagram

which is explained below:

$$\begin{array}{ccccccc}
 \mathcal{E}_W & \xrightarrow{q} & \mathcal{E}_Y & & & & \\
 & \searrow 0 & \downarrow \xi & \searrow p & & \searrow 0 & \\
 & & \mathcal{E}_U & & & & \\
 & & \downarrow \eta & \searrow \beta\eta & & & \\
 0 & \longrightarrow & T_1 & \xrightarrow{\alpha} & T & \longrightarrow & T_2 \longrightarrow 0
 \end{array}$$

By the effaceability of  $T_2$  there exists a deflation  $p : Y \twoheadrightarrow U$  such that  $\beta\eta p = 0$  and so  $\eta p = T(p)(\eta)$  factors through  $\alpha$  via  $\xi \in \text{fp-}\mathcal{E}(\mathcal{E}_Y, T_1)$ . Now, since  $T_1$  is effaceable, there exists  $q : W \twoheadrightarrow Y$  such that  $\xi q = T_1(q)(\xi) = 0$  and so also  $0 = \alpha\xi q = \eta pq = T(pq)(\eta)$ . We remark that  $pq$  is a deflation since it is a composition of two deflations and so the previous construction proves that  $T$  is effaceable.

Let us prove that  $\text{eff-}\mathcal{E}_x\mathcal{E}$  is closed under admissible subobjects and admissible quotients. Let  $0 \rightarrow T_1 \xrightarrow{\alpha} T \xrightarrow{\beta} T_2 \rightarrow 0$  be a conflation in  $\text{fp-}\mathcal{E}$  with  $T \in \text{eff-}\mathcal{E}_x\mathcal{E}$ . Let  $U \in \mathcal{E}$  and  $\eta \in T_1(U)$  hence there exists a deflation  $p : Y \twoheadrightarrow U$  such that  $\alpha(Y)(T_1(p)(\eta)) = T(p)(\alpha(U)(\eta)) = 0$  which proves that  $T_1(p)(\eta) = 0$  (since  $\alpha(Y)$  is a monomorphism of abelian groups by the Proposition 4.1). Let  $V \in \mathcal{E}$  and  $\xi \in T_2(V) \cong \text{fp-}\mathcal{E}(\mathcal{E}_V, T_2)$ . Hence by the projectivity of  $\mathcal{E}_V$  there exists  $\sigma : \mathcal{E}_V \rightarrow T$  and by the effaceability of  $T$  there exists  $q : W \twoheadrightarrow V$  such that the following diagram commutes:

$$\begin{array}{ccc}
 & & 0 \\
 & \nearrow & \searrow \\
 & \mathcal{E}_V & \xleftarrow{q} \mathcal{E}_W \\
 \downarrow \sigma & \searrow \xi & \\
 T & \xrightarrow{\beta} & T_2
 \end{array}$$

which proves that  $\xi q = T_2(q)(\xi) = 0$  and so  $T_2$  is effaceable too.

Now let  $f : U \rightarrow A$  with  $U \in \text{fp-}\mathcal{E}$  and  $A \in \text{eff-}\mathcal{E}_x\mathcal{E}$ ; let  $\mathcal{E}_{U_1} \xrightarrow{h} \mathcal{E}_{U_2} \rightarrow U \rightarrow 0$  (respectively  $\mathcal{E}_{A_1} \xrightarrow{p} \mathcal{E}_{A_2} \rightarrow A \rightarrow 0$ ) a presentation of  $U$  (respectively  $A$  with  $p$  a deflation). Hence we obtain the following commutative diagram and since the pointwise kernel  $\text{Ker}(\pi) = \text{Coker}_{\text{fp-}\mathcal{E}}(r) \in \text{fp-}\mathcal{E}$  we deduce that the sequence  $0 \rightarrow \text{Coker}_{\text{fp-}\mathcal{E}}(r) \rightarrow U \rightarrow \text{Coker}_{\text{fp-}\mathcal{E}}(\pi) \rightarrow 0$  is a conflation

$$\begin{array}{ccccc}
 \mathcal{E}_{U_1} & \xrightarrow{r} & \mathcal{E}_{U_2 \times_{A_2} A_1} & \xrightarrow{\quad} & \mathcal{E}_{A_1} \\
 & \searrow h & \downarrow q & & \downarrow p \\
 & & \mathcal{E}_{U_2} & \xrightarrow{\quad} & \mathcal{E}_{A_2} \\
 & & \downarrow & & \downarrow \\
 & & U & \xrightarrow{f} & A \\
 & & \searrow \pi & \nearrow g & \\
 & & & \text{Coker}_{\text{fp-}\mathcal{E}}(q) & 
 \end{array}$$

When  $\mathcal{A}$  is an abelian category condition (1) and (2) prove that  $\text{eff-}\mathcal{E}_x\mathcal{A}$  is a Serre subcategory of  $\text{coh-}\mathcal{A}$ . The left statement holds true by duality (in  $\mathcal{E}^\circ$ ).  $\square$

**Remark 4.8.** By Proposition 4.7 we can apply Lemma 4.4 thus performing the quotient of  $\text{fp-}\mathcal{E}$  with respect to  $\text{eff-}\mathcal{E}_x\mathcal{E}$ :

$$R_{(\mathcal{E}, \mathcal{E}_x)} : \mathcal{E} \longrightarrow \frac{\text{fp-}\mathcal{E}}{\text{eff-}\mathcal{E}_x\mathcal{E}}$$

the quotient functor is fully faithful since given  $E_1, E_2 \in \mathcal{E}$  (which are monofunctors) we have:

$$\frac{\text{fp-}\mathcal{E}}{\text{eff-}\mathcal{E}_x\mathcal{E}}(\mathcal{E}_{E_1}, \mathcal{E}_{E_2}) \cong \lim_{i: X \rightarrow \mathcal{E}_{E_1}} \text{fp-}\mathcal{E}(X, \mathcal{E}_{E_2})$$

where  $i$  is an inflation in  $\text{fp-}\mathcal{E}$  such that its cokernel is effaceable and so by Remark 4.6 there exists  $\pi : E_3 \twoheadrightarrow X$  such that  $p := i\pi$  is a deflation in  $\mathcal{E}$ . Let us denote by  $\text{Ker}_{\mathcal{E}}(p) \xrightarrow{\alpha} E_3$  its kernel, hence  $\text{Coker}_{\text{fp-}\mathcal{E}}(\alpha) \cong X$  while  $\text{Coker}_{\mathcal{E}}(\alpha) \cong E_1$  and hence any morphism from  $X \rightarrow \mathcal{E}_{E_2}$  extends to a unique morphism  $\mathcal{E}_{E_1} \rightarrow \mathcal{E}_{E_2}$  in  $\text{fp-}\mathcal{E}$ .

## 5. $n$ -COHERENT CATEGORIES

In the previous sections we have seen that the main difference between the 1 and the 2 setting is the need of the introduction of Quillen exact structures. The passage from the 2 case to the general  $n$  case with  $n \geq 3$  requires a new technicality due to the possible absence of kernels and cokernels.

So let  $(\mathcal{E}, \mathcal{E}_x)$  be a projectively complete category endowed with a Quillen exact structure. We are looking for a definition of  $n$ -quasi-abelian category which permits us to associate to  $(\mathcal{E}, \mathcal{E}_x)$  a  $n$ -tilting pair of  $t$ -structures on  $D(\mathcal{E}, \mathcal{E}_x) := K(\mathcal{E})/\mathcal{N}_{\mathcal{E}_x}$ . By Proposition 2.5 we know that if these  $t$ -structures exist they are the left and right  $t$ -structures:

$$\begin{aligned} \mathcal{LD}_{(\mathcal{E}, \mathcal{E}_x)}^{\leq 0} &:= \{X^\bullet \in K(\mathcal{E}) \mid X^\bullet \cong E_{\leq 0}^\bullet \text{ in } D(\mathcal{E}, \mathcal{E}_x) \text{ with } E_{\leq 0}^\bullet \in K^{\leq 0}(\mathcal{E})\} \\ \mathcal{RD}_{(\mathcal{E}, \mathcal{E}_x)}^{\geq 1} &:= \{X^\bullet \in K(\mathcal{E}) \mid X^\bullet \cong E_{\geq 1}^\bullet \text{ in } D(\mathcal{E}, \mathcal{E}_x) \text{ with } E_{\geq 1}^\bullet \in K^{\geq 1}(\mathcal{E})\}. \end{aligned}$$

In the following we will use the notions of coherent functor, coherent category (Definition B.11) weak kernels and cokernels; we refer to Appendix B for more details. First of all we study the case of  $(\mathcal{E}, \mathcal{E}_{x_{\text{split}}})$  endowed with its minimal Quillen exact structure so that  $D(\mathcal{E}, \mathcal{E}_{x_{\text{split}}}) = K(\mathcal{E})$ .

**Proposition 5.1.** *The followings hold:*

- (1) *the class  $\mathcal{LK}_{\mathcal{E}}^{\leq 0} := K^{\leq 0}(\mathcal{E})$  is an aisle if and only if  $\mathcal{E}$  is right coherent;*
- (2) *the class  $\mathcal{RK}_{\mathcal{E}}^{\geq 1} := K^{\geq 1}(\mathcal{E})$  is a co-aisle if and only if  $\mathcal{E}$  is left coherent.*

*If  $\mathcal{E}$  is a coherent category we have  $\mathcal{LK}(\mathcal{E}) \cong \text{coh-}\mathcal{E}$  while  $\mathcal{RK}(\mathcal{E}) \cong (\mathcal{E}\text{-coh})^\circ$  and  $\mathcal{RK}_{\mathcal{E}}^{\leq -n} \subseteq \mathcal{LK}_{\mathcal{E}}^{\leq 0} \subseteq \mathcal{RK}_{\mathcal{E}}^{\leq 0}$  if and only if  $\text{coh-}\mathcal{E}$  (or equivalently  $\mathcal{E}\text{-coh}$ ) has projective dimension  $n$ .*

*Proof.* Statement (2) is dual to (1). Let us recall that by Proposition B.17  $\mathcal{E}$  is right coherent if and only if it admits weak kernels. Let us suppose that  $K^{\leq 0}(\mathcal{E})$  is an aisle (we denote by  $\tau_{\mathcal{E}}^{\leq 0}$  its truncation functor). Let  $d : E_0 \rightarrow E_1$  be a morphism in  $\mathcal{E}$  and let us regard it as a complex  $E^\bullet := E_0 \xrightarrow{d} E_1$  (with  $E^0$  placed in degree 0).

The universal property in  $K(\mathcal{E})$  of the truncation  $[\cdots \rightarrow K^{-1} \rightarrow \overset{\bullet}{K^0} \rightarrow 0 \rightarrow \cdots] = \tau_{\mathcal{E}}^{\leq 0}(E^\bullet) \xrightarrow{\alpha^\bullet} E^\bullet$  proves that  $(K^0, \alpha^0)$  is a weak kernel for  $d$ . On the other side if  $\mathcal{E}$  is right coherent the Freyd category of (contravariant) finitely presented functor is

abelian  $\text{fp-}\mathcal{E} = \text{coh-}\mathcal{E}$  and  $\mathcal{E}$  coincides with the class of projective objects in  $\text{coh-}\mathcal{E}$ ; hence we can define the truncation functor as the following composition:

$$\tau_{\mathcal{E}}^{\leq 0} : K(\mathcal{E}) \longrightarrow D(\text{coh-}\mathcal{E}) \xrightarrow{\delta^{\leq 0}} D^{\leq 0}(\text{coh-}\mathcal{E}) \cong K^{\leq 0}(\mathcal{E}).$$

In particular if  $E^\bullet = [E^0 \xrightarrow{d} E^1]$  and  $\tau^{\leq 0}(E^\bullet) := [\cdots \rightarrow K^{-1} \xrightarrow{d_K^{-1}} K^0 \rightarrow 0 \rightarrow \cdots] \xrightarrow{\alpha^\bullet} E^\bullet$  we have that  $K^\bullet \rightarrow \text{Ker}_{\text{coh-}\mathcal{E}}(d)$  gives a resolution of  $\text{Ker}_{\text{coh-}\mathcal{E}}(d)$  with projective objects in  $\text{coh-}\mathcal{E}$  (and hence in  $\mathcal{E}$ ). Therefore  $(K^0, \alpha^0)$  is a weak kernel for  $d$ ,  $(K^{-1}, d_K^{-1})$  is a weak kernel of  $\alpha^0$  and  $K^{i-1}$  is a weak kernel of  $d_K^i$  for any  $i \leq -2$ .

Let us suppose that  $\mathcal{E}$  is right coherent. Any object  $X^\bullet \in \mathcal{LK}(\mathcal{E})$  is isomorphic to  $X^\bullet \cong \tau_{\mathcal{E}}^{\geq 0} \tau_{\mathcal{E}}^{\leq 0} X^\bullet$  (see 1.12 for the definitions of  $\tau_{\mathcal{E}}^{\leq 0}, \tau_{\mathcal{E}}^{\geq 0}$ ), so it can be represented as a complex  $C(d) := [\text{Ker } d \rightarrow X \xrightarrow{d} Y]$  (see notation C.3) and morphisms between two such complexes are morphisms in  $K(\mathcal{E})$ . Hence the functor

$$\begin{aligned} A : \mathcal{LK}(\mathcal{E}) &\longrightarrow \text{coh-}\mathcal{E} \\ C(d) &\longmapsto \text{Coker}_{\text{coh-}\mathcal{E}}(d) \end{aligned}$$

is well defined and faithful (since a morphism gives rise to the 0 morphism between the cokernels if and only if it is homotopic to 0). It is essentially surjective since any coherent functor  $F \in \text{coh-}\mathcal{E}$  is finitely presented thus there exists a presentation  $h_X \xrightarrow{d} h_Y \rightarrow F \rightarrow 0$  (where  $h_X := \mathcal{E}(-, X)$  is the functor represented by  $X$ , hence by the Yoneda Lemma  $d \in \mathcal{E}(X, Y)$ ). It is fully faithful since  $\mathcal{E}$  coincides with the class of projective objects both in  $\mathcal{LH}(\mathcal{E})$  and  $\text{coh-}\mathcal{E}$ , so any morphism  $f \in \text{coh-}\mathcal{E}(\text{Coker}_{\text{coh-}\mathcal{E}}(d), \text{Coker}_{\text{coh-}\mathcal{E}}(d'))$  lifts to a morphism  $\mathcal{LH}(\mathcal{E})(C(d), C(d'))$ .

The category  $\text{coh-}\mathcal{E}$  has finite projective dimension  $n$  if and only if given any  $E^\bullet \in K^{\geq 0}(\mathcal{E})$  the kernel  $\text{Ker}_{\text{coh-}\mathcal{E}}(d_E^0)$  admits a resolution of length at most  $n-2$  (since  $0 \rightarrow \text{Ker}_{\text{coh-}\mathcal{E}}(d_E^0) \rightarrow E^0 \rightarrow E^1 \rightarrow \text{Coker}_{\text{coh-}\mathcal{E}}(d_E^0) \rightarrow 0$  is exact and any projective resolution of  $\text{Coker}_{\text{coh-}\mathcal{E}}(d_E^0)$  has at most length  $n$ ). Since for any  $X^\bullet \in K(\mathcal{E})$  we have (see 1.12)  $\tau_{\mathcal{E}}^{\geq 1} X^\bullet \cong \tau_{\mathcal{E}}^{\geq 1} X^{\geq 0} \subseteq K^{\geq -n+2}(\mathcal{E})$  we deduce that the previous holds if and only if  $\mathcal{LK}_{\mathcal{E}}^{\geq 1} \subseteq \mathcal{RK}_{\mathcal{E}}^{\geq -n+1}$  and hence  $\mathcal{RK}_{\mathcal{E}}^{\leq -n} \subseteq \mathcal{LK}_{\mathcal{E}}^{\leq 0} \subseteq \mathcal{RK}_{\mathcal{E}}^{\leq 0}$ . In this case the right heart  $\mathcal{RK}(\mathcal{E}) \subseteq K^{[0, n]}(\mathcal{E})$  so any  $N \in (\mathcal{E}\text{-coh})^\circ \cong \mathcal{RK}(\mathcal{E})$  can be represented as a complex  $[E^0 \xrightarrow{d_E^0} \cdots \rightarrow E^n]$  exact at any  $i > 0$  in  $(\mathcal{E}\text{-coh})^\circ$  and such that  $\text{Ker}_{(\mathcal{E}\text{-coh})^\circ} d_E^0 = N$  which proves that  $\mathcal{E}\text{-coh}$  has projective dimension  $n$ .

In this case  $n$  is called the *global dimension* of  $\mathcal{E}$ .  $\square$

**Definition 5.2.** A coherent category of global dimension at most  $n$  will be said *n-coherent*. For example the category  $\text{proj-}R$  of projective (right) modules of finite type on a coherent ring  $R$  with global dimension  $n$  is *n-coherent*.

**Remark 5.3.** A right Freyd inclusion of a *n-coherent* category is the canonical functor of  $\mathcal{E} \hookrightarrow (\mathcal{E}\text{-coh})^\circ$  with  $\mathcal{E}$  a *n-coherent* category and this is the analog of a *n-tilting* torsion class in this setting. Let  $\mathcal{A} := (\mathcal{E}\text{-coh})^\circ$ ; in this case one can characterize this embedding functor as follows:

- (1)  $\mathcal{E}$  is a *n-coherent* category and it coincides with the class of injectives in  $\mathcal{A}$ ;
- (2)  $\mathcal{E}$  is fully faithful and cogenerating in  $\mathcal{A}$ ;
- (3) for any exact sequence in  $\mathcal{A}$ :

$$0 \longrightarrow A \longrightarrow X_{-n+1} \xrightarrow{d_X^{-n+1}} \cdots \xrightarrow{d_X^{-1}} X_0 \xrightarrow{d_X^0} B \longrightarrow 0$$

with  $X_i \in \mathcal{E}$  for any  $-n+1 \leq i \leq 0$  and  $A, B \in \mathcal{A}$  we have  $B \in \mathcal{E}$ .

In this case the square analog to the one of Theorem 6.15 would be

$$\begin{array}{ccc}
 \{n\text{-coherent categories}\} & \longleftrightarrow & \{n\text{-tilting pairs of } t\text{-struc. on } K(\mathcal{E})\} \\
 \uparrow & & \uparrow \\
 \mathcal{E} = \mathcal{RK}(\mathcal{E}) \cap \mathcal{LK}(\mathcal{E}) & \longleftrightarrow & (\mathcal{RK}_{\mathcal{E}}, \mathcal{LK}_{\mathcal{E}}) \text{ on } \mathcal{C} = K(\mathcal{E}) \\
 \downarrow & & \downarrow \\
 \{\text{left Freyd inclusions of a } n\text{-coherent cat.}\} & \longleftrightarrow & \{\text{right Freyd inclusions of a } n\text{-coherent cat.}\} \\
 \downarrow & & \downarrow \\
 \mathcal{E} \text{ in } (\mathcal{E}\text{-coh})^{\circ} & \longleftrightarrow & \mathcal{E} \text{ in coh-}\mathcal{E}.
 \end{array}$$

## 6. $n$ -QUASI-ABELIAN CATEGORIES VS $n$ -TILTING TORSION PAIRS FOR $n > 2$

This section is devoted to the general notion of  $n$ -quasi-abelian category for a projectively complete exact category  $(\mathcal{E}, \mathcal{E}x)$ . When the exact structure is not the minimal one we will provide in Lemma 6.3 necessary and sufficient conditions for the existence of the left and right  $t$ -structures on  $D(\mathcal{E}, \mathcal{E}x) := K(\mathcal{E})/\mathcal{N}_{\mathcal{E}x}$  and we will describe the left heart  $\mathcal{LH}(\mathcal{E}, \mathcal{E}x)$  (respectively right) in terms of quotient of the Freyd category  $\text{fp-}\mathcal{E}$  (respectively  $(\mathcal{E}\text{-fp})^{\circ}$ ) with respect to its full subcategory of effaceable functors. We need preliminary the following definition:

**Definition 6.1.** Let  $(\mathcal{E}, \mathcal{E}x)$  be a projectively complete category endowed with a Quillen exact structure and  $f : A \rightarrow B$  a morphism in  $\mathcal{E}$ . A  $D(\mathcal{E}, \mathcal{E}x)$ -kernel of  $f$  is a map  $i : K \rightarrow A$  in  $\mathcal{E}$  such that  $f \circ i = 0$  and for any  $j : X \rightarrow A$  such that  $f \circ j = 0$  there exist (possibly many) a deflation  $\pi : N \twoheadrightarrow X$  and a map  $k : W \rightarrow K$  such that  $j\pi = ik$ :

$$\begin{array}{ccccc}
 N & \cdots\cdots\cdots \twoheadrightarrow & X & & \\
 \vdots & & \downarrow j & \searrow 0 & \\
 K & \xrightarrow{i} & A & \xrightarrow{f} & B \\
 & \searrow & \downarrow & \nearrow & \\
 & & 0 & & 
 \end{array}$$

The category  $(\mathcal{E}, \mathcal{E}x)$  has  $D(\mathcal{E}, \mathcal{E}x)$ -pull-back squares if given any pair  $f_i : X_i \rightarrow Y$  with  $i = 1, 2$  there exists an object  $Z$  with the dashed arrows such that any commutative diagram of this type can be completed with (not necessarily unique) dotted arrows:

$$(2) \quad \begin{array}{ccccccc}
 & & & & & & \\
 & & & & & & \\
 W & \xleftarrow{\pi} & N & \cdots\cdots\cdots \xrightarrow{k} & Z & \xrightarrow{g_1} & X_1 \\
 & \searrow & & & \downarrow g_2 & & \downarrow f_1 \\
 & & & & X_2 & \xrightarrow{f_2} & Y.
 \end{array}$$

Passing throughout the opposite category one obtain the dual notion of  $D(\mathcal{E}, \mathcal{E}x)$ -cokernel and  $D(\mathcal{E}, \mathcal{E}x)$ -push-out square.

**Remark 6.2.** When the Quillen exact structure is the minimal one (split short exact sequences) we have  $D(\mathcal{E}, \mathcal{E}x_{\text{split}}) = K(\mathcal{E})$ . The previous definition coincides with the notions *weak kernel* and *weak pull-back square* (see Definition B.16).

If  $\mathcal{E}$  admits weak kernels (equivalently it is right coherent) hence it admits  $D(\mathcal{E}, \mathcal{E}x)$ -kernels for any Quillen exact structure on  $\mathcal{E}$  since any weak kernel is



also a  $D(\mathcal{E}, \mathcal{E}x)$ -kernel. More generally if  $\mathcal{E}$  admits  $D(\mathcal{E}, \mathcal{E}x)$ -kernels hence for any other Quillen exact structure  $\overline{\mathcal{E}x}$  containing the conflations of  $\mathcal{E}x$  we have that  $\mathcal{E}$  admits  $D(\mathcal{E}, \overline{\mathcal{E}x})$ -kernels. The problem is that it seems to us that a category  $\mathcal{E}$  could admit  $D(\mathcal{E}, \mathcal{E}x)$ -kernels (for example with respect its maximal Quillen exact structure) without admitting weak kernels.

**Lemma 6.3.** *Let  $(\mathcal{E}, \mathcal{E}x)$  be a projectively complete category endowed with a Quillen exact structure.*

- (1) *The subcategory  $\mathcal{LD}_{(\mathcal{E}, \mathcal{E}x)}^{\leq 0}$  is an aisle in  $D(\mathcal{E}, \mathcal{E}x)$  if and only if  $\mathcal{E}$  has  $D(\mathcal{E}, \mathcal{E}x)$ -kernels.*
- (2) *The subcategory  $\mathcal{RD}_{(\mathcal{E}, \mathcal{E}x)}^{\geq 1}$  is a co-aisle in  $D(\mathcal{E}, \mathcal{E}x)$  if and only if  $\mathcal{E}$  has  $D(\mathcal{E}, \mathcal{E}x)$ -cokernels.*

*If the previous conditions are satisfied we have that  $\mathcal{E} = \mathcal{LD}_{(\mathcal{E}, \mathcal{E}x)}^{\leq 0} \cap \mathcal{RD}_{(\mathcal{E}, \mathcal{E}x)}^{\geq 0}$  and any object in the heart  $\mathcal{LH}(\mathcal{E}, \mathcal{E}x)$  can be represented as a complex  $K^\bullet \in K^{\leq 0}(\mathcal{E})$  such that  $K^i = D(\mathcal{E}, \mathcal{E}x)$ -kernel of  $d_K^{i+1}$  for any  $i \leq -2$ . Dually objects in  $\mathcal{RH}(\mathcal{E}, \mathcal{E}x)$  are complexes  $C^\bullet \in K^{\geq 0}(\mathcal{E})$  such that  $C^i = D(\mathcal{E}, \mathcal{E}x)$ -cokernel of  $d_C^{i-2}$  for any  $i \geq 2$ .*

*Proof.* (1). Let us suppose that  $\mathcal{LD}_{(\mathcal{E}, \mathcal{E}x)}^{\leq 0}$  is an aisle in  $D(\mathcal{E}, \mathcal{E}x)$ . Hence the inclusion functor  $i_{\leq 0} : \mathcal{LD}_{(\mathcal{E}, \mathcal{E}x)}^{\leq 0} \hookrightarrow D(\mathcal{E}, \mathcal{E}x)$  admits a right adjoint  $\tau_{\mathcal{L}}^{\leq 0} : D(\mathcal{E}, \mathcal{E}x) \rightarrow \mathcal{LD}_{(\mathcal{E}, \mathcal{E}x)}^{\leq 0}$  (by abuse of notation we will denote by  $\tau_{\mathcal{L}}^{\leq 0}$  also the composition  $i_{\leq 0} \tau_{\mathcal{L}}^{\leq 0}$ ). Let  $f : A \rightarrow B$  be a morphism in  $\mathcal{E}$  and let regard it as a complex:  $M^\bullet := [\overset{\bullet}{A} \xrightarrow{f} B] \in K^{\geq 0}(\mathcal{E})$  (with  $A$  in degree 0,  $B$  in degree 1 and 0 otherwise). Let  $\alpha : \tau_{\mathcal{L}}^{\leq 0}(M^\bullet) \rightarrow M^\bullet$  (morphism in  $D(\mathcal{E}, \mathcal{E}x)$ ) be the co-unit of the adjunction and  $\tau_{\mathcal{L}}^{\leq 0}(M^\bullet) = [\cdots \rightarrow K^{-1} \rightarrow \overset{\bullet}{K}^0] \in K^{\leq 0}(\mathcal{E})$ . By Lemma A.6 the morphism  $\alpha$  is a morphism in  $K(\mathcal{E})$  (since  $\tau_{\mathcal{L}}^{\leq 0}(M^\bullet) \in K^{\leq 0}(\mathcal{E})$  while  $M^\bullet \in K^{\geq 0}(\mathcal{E})$ ) and let denote by  $i := \alpha^0 : K^0 \rightarrow A$ . Let us prove that  $K^0 \xrightarrow{i} A$  is a  $D(\mathcal{E}, \mathcal{E}x)$ -kernel for  $f$ . Let consider  $j : X \rightarrow A$  a morphism in  $\mathcal{E}$  such that  $fj = 0$ , hence  $j$  induces a morphism  $X[0] \xrightarrow{j} M^\bullet$  in  $D(\mathcal{E}, \mathcal{E}x)$ . Since  $X[0] \in \mathcal{LD}_{(\mathcal{E}, \mathcal{E}x)}^{\leq 0}$  there exists a unique morphism  $\beta : X[0] \rightarrow \tau_{\mathcal{L}}^{\leq 0}(M^\bullet)$  in  $D(\mathcal{E}, \mathcal{E}x)$  such that  $\alpha\beta = j$  i.e. there exists a resolution  $\cdots \rightarrow N^{-1} \rightarrow N^0 \xrightarrow{\pi} X \rightarrow 0$  (which is a complex in the null system  $\mathcal{N}_{\mathcal{E}x}$ ) and a morphism of complexes  $k^\bullet : N^\bullet \rightarrow \tau_{\mathcal{L}}^{\leq 0}(M^\bullet)$  such that  $j\pi : N^0 \xrightarrow{k^0} K^0 \xrightarrow{i} A$  which proves that  $\mathcal{E}$  has  $D(\mathcal{E}, \mathcal{E}x)$ -kernels.

On the other side let us suppose that  $\mathcal{E}$  has  $D(\mathcal{E}, \mathcal{E}x)$ -kernels. Since  $\mathcal{LD}_{(\mathcal{E}, \mathcal{E}x)}^{\leq 0}$  is a full subcategory of  $D(\mathcal{E}, \mathcal{E}x)$  closed by [1] and stable by extensions we have to prove that the inclusion functor  $i_{\leq 0} : \mathcal{LD}_{(\mathcal{E}, \mathcal{E}x)}^{\leq 0} \hookrightarrow D(\mathcal{E}, \mathcal{E}x)$  admits a right adjoint  $\tau_{\mathcal{L}}^{\leq 0} : D(\mathcal{E}, \mathcal{E}x) \rightarrow \mathcal{LD}_{(\mathcal{E}, \mathcal{E}x)}^{\leq 0}$ . Let first compute the  $\tau_{\mathcal{L}}^{\leq 0}(L^\bullet)$  in the case  $L^\bullet := [\overset{\bullet}{L}^0 \xrightarrow{d_L^0} L^1 \rightarrow L^2 \rightarrow \cdots] \in K^{\geq 0}(\mathcal{E})$  (since  $K^{\geq 0}(\mathcal{E}) \subseteq \mathcal{LD}_{(\mathcal{E}, \mathcal{E}x)}^{\geq 0}$  in this case  $\tau_{\mathcal{L}}^{\leq 0}(L^\bullet) \in \mathcal{LH}(\mathcal{E}, \mathcal{E}x)$ ). Let  $K^0 \xrightarrow{i} L^0$  be a  $D(\mathcal{E}, \mathcal{E}x)$ -kernel of  $d_L^0$ ,  $K^{-1} \xrightarrow{d_K^{-1}} K^0$  a  $D(\mathcal{E}, \mathcal{E}x)$ -kernel of  $i$  and recursively let  $K^{-i-1} \xrightarrow{d_K^{-i-1}} K^{-i}$  be a  $D(\mathcal{E}, \mathcal{E}x)$ -kernel of  $d_K^{-i}$  with  $i \geq 1$ . We pose  $\tau_{\mathcal{L}}^{\leq 0}(L^\bullet) := [\cdots \xrightarrow{d_K^{-2}} K^{-2} \xrightarrow{d_K^{-1}} K^{-1} \xrightarrow{d_K^0} K^0] \in K^{\leq 0}(\mathcal{E})$  with  $h : K^\bullet \rightarrow L^\bullet$  the morphism of complexes induced by  $i : K^0 \rightarrow L^0$ . For any other complex  $X^\bullet \in K^{\leq 0}(\mathcal{E})$  we have (by Lemma A.6)  $D(\mathcal{E}, \mathcal{E}x)(X^\bullet, L^\bullet) \cong K(\mathcal{E})(X^\bullet, L^\bullet) \cong D(\mathcal{E}, \mathcal{E}x)(X^\bullet, K^\bullet)$  because

for any  $\alpha \in K(\mathcal{E})(X^\bullet, L^\bullet)$  by definition of  $D(\mathcal{E}, \mathcal{E}x)$ -kernel we can construct the following commutative diagram:

with  $\pi : N^\bullet \rightarrow X^\bullet$  an isomorphism in  $D(\mathcal{E}, \mathcal{E}x)$ ,  $k^\bullet : N^\bullet \rightarrow K^\bullet$  such that  $\alpha\pi = hk$ . If  $\alpha = 0$  the previous construction produces a  $k^\bullet = 0$  in  $D(\mathcal{E}, \mathcal{E}x)$  (since  $d_k^{-1}$  is the  $D(\mathcal{E}, \mathcal{E}x)$ -kernel of  $i$  and, if  $\alpha = 0$ ,  $ik^0 = 0$  there exists  $p^0 : W^0 \rightarrow N^0$  and  $s^0 : W^0 \rightarrow K^{-1}$  such that  $k^0 p^0 = d_k^{-1} s^0$  iterating these  $s^i$  produce a homotopy).

Let  $E^\bullet := [\cdots \rightarrow E^{-1} \rightarrow E^0 \rightarrow E^1 \rightarrow \cdots]$  be a complex in  $K(\mathcal{E})$ , and let consider the following commutative diagram whose rows and columns are distinguished triangles and let put by definition  $\tau_{\mathcal{L}}^{\leq 0}(E^\bullet)$  to be the mapping cone in  $K(\mathcal{E})$  of the morphism  $\tau_{\mathcal{L}}^{\leq 0}(d_{E^\bullet}^{-1})$ :

$$\begin{array}{ccccc}
 (3) & E^{\leq -1}[-1] & \xrightarrow{id} & E^{\leq -1}[-1] & \longrightarrow & 0 & \xrightarrow{+} & \longrightarrow \\
 & \downarrow \tau^{\leq 0}(d_{E^\bullet}^{-1}) & & \downarrow d_{X^\bullet}^{-1} & & \downarrow & & \\
 & \tau_{\mathcal{L}}^{\leq 0}(E^{\geq 0}) & \longrightarrow & E^{\geq 0} & \longrightarrow & \tau_{\mathcal{L}}^{\geq 1}(E^{\geq 0}) & \xrightarrow{+} & \longrightarrow \\
 & \downarrow & & \downarrow & & \downarrow \cong & & \\
 & \tau_{\mathcal{L}}^{\leq 0}(E^\bullet) & \longrightarrow & E^\bullet & \longrightarrow & \tau_{\mathcal{L}}^{\geq 1}(E^\bullet) & \xrightarrow{+} & \longrightarrow \\
 & \downarrow + & & \downarrow + & & \downarrow + & & \\
 & & & & & & & 
 \end{array}$$

For any  $X^\bullet \in K^{\leq 0}(\mathcal{E})$  we have that  $D(\mathcal{E}, \mathcal{E}x)(X^\bullet, E^\bullet) \cong D(\mathcal{E}, \mathcal{E}x)(X^\bullet, \tau_{\mathcal{L}}^{\leq 0}(E^\bullet))$  since  $D(\mathcal{E}, \mathcal{E}x)(X^\bullet, \tau_{\mathcal{L}}^{\geq 1}(E^{\geq 0})) = 0$ . An object in the heart  $\mathcal{LH}(\mathcal{E}, \mathcal{E}x)$  can be represented as a complex  $K^\bullet \in K^{\leq 0}(\mathcal{E})$  such that  $\tau_{\mathcal{L}}^{\leq -1}(K^\bullet) \in \mathcal{N}_{\mathcal{E}x}$  and so  $K^i = D(\mathcal{E}, \mathcal{E}x)$ -kernel of  $d_K^{i+1}$  for any  $i \leq -2$ . Statement (2) is dual to (1).

If  $\mathcal{E}$  admits  $D(\mathcal{E}, \mathcal{E}x)$ -kernels and cokernels, let denote by  $\tau_{\mathcal{L}}^{\leq 0}$  the truncation functor with respect to  $\mathcal{LD}_{(\mathcal{E}, \mathcal{E}x)}^{\leq 0}$  while  $\delta_{\mathcal{R}}^{\geq 1}$  the one with respect to  $\mathcal{RD}_{(\mathcal{E}, \mathcal{E}x)}^{\geq 1}$ . Hence  $E^\bullet \in \mathcal{LD}_{(\mathcal{E}, \mathcal{E}x)}^{\leq 0} \cap \mathcal{RD}_{(\mathcal{E}, \mathcal{E}x)}^{\geq 0}$  if and only if the composition  $\gamma : K^\bullet := \tau_{\mathcal{L}}^{\leq 0} E^\bullet \rightarrow E^\bullet \rightarrow \delta_{\mathcal{R}}^{\geq 0}(E^\bullet) =: C^\bullet$  is an isomorphism in  $D(\mathcal{E}, \mathcal{E}x)$  i.e.; if and only if the mapping cone

$M(\gamma) \in \mathcal{N}_{\mathcal{E}x}$ :

$$\begin{array}{ccccccc}
 K^\bullet : & \cdots & \longrightarrow & K^{-1} & \xrightarrow{d_K^{-1}} & K^0 & \longrightarrow 0 \longrightarrow \cdots \\
 \downarrow \gamma & & & \downarrow & & \downarrow \gamma^0 & \\
 C^\bullet : & \cdots & \longrightarrow & 0 & \longrightarrow & C^0 & \xrightarrow{d_C^0} C^1 \xrightarrow{d_C^1} \cdots \\
 \downarrow & & & \downarrow & & \downarrow & \\
 M(\gamma) : & K^{-1} & \xrightarrow{d_K^{-1}} & K^0 & \xrightarrow{\gamma^0} & C^0 & \xrightarrow{d_C^0} C^1 \xrightarrow{d_C^1} \cdots \\
 & \searrow & & \swarrow & & \searrow & \swarrow \\
 & & W^{-1} & & W^0 & & W^1 & & W^2 & \cdots
 \end{array}$$

this proves that  $K^\bullet \cong W^0[0] \in \mathcal{E}$ .  $\square$

**Lemma 6.4.** *Let us suppose that  $(\mathcal{E}, \mathcal{E}x)$  admits  $D(\mathcal{E}, \mathcal{E}x)$ -kernels and cokernels. Hence  $\mathcal{RD}_{(\mathcal{E}, \mathcal{E}x)}^{\leq -n} \subseteq \mathcal{LD}_{(\mathcal{E}, \mathcal{E}x)}^{\leq 0} \subseteq \mathcal{RD}_{(\mathcal{E}, \mathcal{E}x)}^{\leq 0}$  (with  $n \geq 2$ ) if and only if one of the following equivalent conditions holds:*

- given any complex  $K^{[-n+1, 0]} := K^{-n+1} \xrightarrow{d_K^{-n+1}} K^{-n+2} \longrightarrow \cdots \xrightarrow{d_K^{-1}} K^0$  with  $K^i = D(\mathcal{E}, \mathcal{E}x)$ -kernel of  $d_K^{i+1}$  for any  $i \leq -2$ , the morphism  $d_K^{-n+1}$  has a kernel in  $\mathcal{E}$ ;
- given any complex  $C^{[-n+1, 0]} := C^{-n+1} \xrightarrow{d_C^{-n+1}} C^{-n+2} \longrightarrow \cdots \xrightarrow{d_C^{-1}} C^0$  with  $C^i = D(\mathcal{E}, \mathcal{E}x)$ -cokernel of  $d_C^{i-2}$  for any  $i \geq -n-1$ , the morphism  $d_C^{-1}$  has a cokernel in  $\mathcal{E}$ .

In this case the pair  $(\mathcal{RD}_{(\mathcal{E}, \mathcal{E}x)}, \mathcal{LD}_{(\mathcal{E}, \mathcal{E}x)})$  is a  $n$ -tilting pair of  $t$ -structures on  $D(\mathcal{E}, \mathcal{E}x)$ .

*Proof.* If  $n \geq 2$  and  $\mathcal{RD}_{(\mathcal{E}, \mathcal{E}x)}^{\leq -n} \subseteq \mathcal{LD}_{(\mathcal{E}, \mathcal{E}x)}^{\leq 0} \subseteq \mathcal{RD}_{(\mathcal{E}, \mathcal{E}x)}^{\leq 0}$  the conditions on  $K^\bullet$  assure that  $\tau_{\mathcal{L}}^{\geq -n+2}(K^{[-n+1, 0]}) \cong \tau_{\mathcal{L}}^{\geq 0}(K^{[-n+1, 0]}) \in \mathcal{LH}(\mathcal{E}, \mathcal{E}x) \subseteq \mathcal{RD}_{(\mathcal{E}, \mathcal{E}x)}^{\geq -n}$  and  $K^{[-n+1, 0]} \in \mathcal{RD}_{(\mathcal{E}, \mathcal{E}x)}^{[-n+1, 0]}$ ; so the distinguished triangle

$$\tau_{\mathcal{L}}^{\leq -n+1}(K^{[-n+1, 0]}) \rightarrow K^{[-n+1, 0]} \rightarrow \tau_{\mathcal{L}}^{\geq -n+2}(K^{[-n+1, 0]}) \xrightarrow{\pm}$$

proves that  $\tau_{\mathcal{L}}^{\leq -n+1}(K^{[-n+1, 0]}) \in \mathcal{RD}_{(\mathcal{E}, \mathcal{E}x)}^{\geq -n+1} \cap \mathcal{LD}_{(\mathcal{E}, \mathcal{E}x)}^{\leq -n+1} = \mathcal{E}[n-1]$ . The dual argument proves that also (2) holds true. On the other side if (1) holds true  $\tau_{\mathcal{L}}^{\geq 1} X^\bullet \cong \tau_{\mathcal{L}}^{\geq 1} X^{\geq 0} \subseteq \mathcal{RD}_{(\mathcal{E}, \mathcal{E}x)}^{\geq -n+2}$  and hence  $\mathcal{RD}_{(\mathcal{E}, \mathcal{E}x)}^{\leq -n} \subseteq \mathcal{LD}_{(\mathcal{E}, \mathcal{E}x)}^{\leq 0} \subseteq \mathcal{RD}_{(\mathcal{E}, \mathcal{E}x)}^{\leq 0}$ .

Therefore any object  $K^\bullet \in \mathcal{LH}(\mathcal{E}, \mathcal{E}x)$  can be represented as a complex  $K^\bullet \in K^{\leq 0}(\mathcal{E})$  such that  $\tau_{\mathcal{L}}^{\leq -1}(K^\bullet) \in \mathcal{N}_{\mathcal{E}x}$  and so it can be represented by a complex

$$C(d_K^{-n}, \dots, d_K^{-1}) := [\text{Ker}(d_K^{-n+1}) \xrightarrow{d_K^{-n}} K^{-n+1} \xrightarrow{d_K^{-n+1}} K^{-n+2} \longrightarrow \cdots \xrightarrow{d_K^{-1}} K^0]$$

such that  $K^i = D(\mathcal{E}, \mathcal{E}x)$ -kernel of  $d_K^{i+1}$  for any  $i \leq -2$ . It fits into the short exact sequence in  $\mathcal{LH}(\mathcal{E}, \mathcal{E}x)$  (distinguished triangle in  $D(\mathcal{E}, \mathcal{E}x)$ )

$$\begin{array}{c}
 K^{-1}[0] \\
 \downarrow \\
 0 \longrightarrow C(0, d_K^{-n}, \dots, d_K^{-2}) \longrightarrow K^0[0] \longrightarrow C(d_K^{-n}, \dots, d_K^{-1}) \longrightarrow 0
 \end{array}$$

and hence  $K^{-1}[0] \rightarrow K^0[0] \rightarrow C(d_K^{-n+1}, \dots, d_K^{-1}) \rightarrow 0$  is exact which proves that  $\mathcal{E}$  generates  $\mathcal{LH}(\mathcal{E}, \mathcal{E}x)$ . Hence the full subcategory  $\mathcal{E}$  in  $\mathcal{LH}(\mathcal{E}, \mathcal{E}x)$  satisfies the hypotheses of Proposition C.11 thus proving that  $K(\mathcal{E})/\mathcal{N}_{\mathcal{E}x} \cong D(\mathcal{LH}(\mathcal{E}, \mathcal{E}x))$ . Dually  $K(\mathcal{E})/\mathcal{N}_{\mathcal{E}x} \cong D(\mathcal{RH}(\mathcal{E}, \mathcal{E}x))$ .  $\square$

**Definition 6.5.** A projectively complete category  $(\mathcal{E}, \mathcal{E}x)$  endowed with a Quillen exact structure is called *n-quasi-abelian* (for  $n \geq 2$ ) if it admits  $D(\mathcal{E}, \mathcal{E}x)$ -kernels and  $D(\mathcal{E}, \mathcal{E}x)$ -cokernels and one of the following equivalent conditions holds:

- (1) For any complex  $K^{[-n+1, 0]} := K^{-n+1} \xrightarrow{d_K^{-n+1}} K^{-n+2} \longrightarrow \dots \xrightarrow{d_K^{-1}} K^0$  with  $K^i = D(\mathcal{E}, \mathcal{E}x)$ -kernel of  $d_K^{i+1}$  for any  $i \leq -2$  the morphism  $d_K^{-n+1}$  has a kernel in  $\mathcal{E}$ .
- (2) For any complex  $C^{[-n+1, 0]} := C^{-n+1} \xrightarrow{d_C^{-n+1}} C^{-n+2} \longrightarrow \dots \xrightarrow{d_C^{-1}} C^0$  with  $C^i = D(\mathcal{E}, \mathcal{E}x)$ -cokernel of  $d_C^{i-2}$  for any  $i \geq -n-1$  the morphism  $d_C^{-1}$  has a cokernel in  $\mathcal{E}$ .

Whenever the exact structure is not specified we will consider  $\mathcal{E}$  endowed with its maximal Quillen exact structure.

**Remark 6.6.** Given  $(\mathcal{E}, \mathcal{E}x)$  a  $n$ -quasi-abelian category by Lemma 6.4 we get a  $n$ -tilting pair of  $t$ -structures  $(\mathcal{RD}_{(\mathcal{E}, \mathcal{E}x)}, \mathcal{LD}_{(\mathcal{E}, \mathcal{E}x)})$  on  $D(\mathcal{E}, \mathcal{E}x)$ . On the other side given a  $n$ -tilting pair of  $t$ -structures  $(\mathcal{D}, \mathcal{T})$  on  $\mathcal{C}$  by Proposition 2.5 and Lemma 6.3 the category  $\mathcal{E} = \mathcal{T}^{\leq 0} \cap \mathcal{D}^{\geq 0}$  admits  $D(\mathcal{E}, \mathcal{E}x)$ -kernels and  $D(\mathcal{E}, \mathcal{E}x)$ -cokernels and hence  $\mathcal{E}$  is a  $n$ -quasi-abelian category (with the Quillen exact structure induced by  $\mathcal{C} = D(\mathcal{E}, \mathcal{E}x)$ ).

The consequences of the  $n$ -tilting Theorem (see 2.1) and Definition 3.3 suggest the following  $n$ -level generalization of the notion of 1-tilting torsion class in an abelian category:

**Definition 6.7.** Let  $\mathcal{A}$  be an abelian category. A full subcategory  $\mathcal{E} \hookrightarrow \mathcal{A}$  is a *n-tilting torsion class* if

- (1)  $\mathcal{E}$  cogenerates  $\mathcal{A}$ ;
- (2)  $\mathcal{E}$  is closed under extensions in  $\mathcal{A}$  and hence it is endowed with a Quillen exact structure  $\mathcal{E}x$  whose conflations are sequence in  $\mathcal{E}$  which are exact in  $\mathcal{A}$ ;
- (3)  $(\mathcal{E}, \mathcal{E}x)$  has  $D(\mathcal{E}, \mathcal{E}x)$ -kernels;
- (4) for any exact sequence in  $\mathcal{A}$ :

$$0 \longrightarrow A \longrightarrow X_n \xrightarrow{d_X^n} \dots \xrightarrow{d_X^2} X_1 \longrightarrow B \longrightarrow 0$$

with  $X_i \in \mathcal{E}$  for any  $1 \leq i \leq n$  and  $A, B \in \mathcal{A}$  we have  $B \in \mathcal{E}$ .

Hence any  $n$ -tilting torsion class is also a  $n+1$ -tilting torsion class.

Dually a *n-cotilting torsion-free class* in  $\mathcal{A}$  is a full generating extension closed subcategory  $\mathcal{E}$  of  $\mathcal{A}$  such that  $(\mathcal{E}, \mathcal{E}x)$  has  $D(\mathcal{E}, \mathcal{E}x)$ -cokernels and such that for any exact sequence in  $\mathcal{A}$ :

$$0 \longrightarrow A \longrightarrow Y_1 \xrightarrow{d_Y^1} \dots \longrightarrow Y_{n-1} \xrightarrow{d_Y^{n-1}} Y_n \longrightarrow B \longrightarrow 0$$

with  $Y_i \in \mathcal{E}$  for any  $1 \leq i \leq n$  and  $A, B \in \mathcal{A}$  we have  $A \in \mathcal{E}$ .

**Remark 6.8.** Let  $(\mathcal{E}, \mathcal{E}x)$  be a  $n$ -quasi-abelian category, hence  $\mathcal{E}$  is a  $n$ -tilting torsion class in  $\mathcal{RH}(\mathcal{E}, \mathcal{E}x)$ . The following Theorem proves that for any  $n$ -tilting torsion class  $\mathcal{E}$  in  $\mathcal{A}$  we have  $\mathcal{A} \cong \mathcal{RH}(\mathcal{E}, \mathcal{E}x)$ .

**Theorem 6.9.** *Any  $n$ -tilting torsion class  $\mathcal{E}$  in  $\mathcal{A}$ , endowed with the Quillen exact structure induced by  $\mathcal{A}$ , is a  $n$ -quasi-abelian category. Moreover a sequence*

*$C^{-n+1} \xrightarrow{d_C^{-n+1}} C^{-n+2} \longrightarrow \dots \xrightarrow{d_C^{-1}} C^0$  is exact in  $\mathcal{A}$  if and only if  $C^i = D(\mathcal{E}, \mathcal{E}x)$ -cokernel of  $d_C^{i-2}$  for any  $i \geq -n-1$  and the morphism  $d_C^{-1}$  has a cokernel in  $\mathcal{E}$ . Therefore  $\mathcal{A} \cong \mathcal{RH}(\mathcal{E}, \mathcal{E}x)$  and hence  $\frac{K(\mathcal{E})}{N_{\mathcal{E}x}} \xrightarrow{\cong} D(\mathcal{A})$ .*

*Proof.* By point (3) of Definition 6.5  $\mathcal{E}$  admits  $D(\mathcal{E}, \mathcal{E}x)$ -kernels. Let  $K^0 \xrightarrow{d} K^1$  a morphism in  $\mathcal{E}$ . Hence, since  $\mathcal{E}$  cogenerates  $\mathcal{A}$  (by point (1) of Definition 6.5), we can find an injection  $\text{Coker}_{\mathcal{A}}(d) \hookrightarrow K^2$ . Let us prove that  $K^2$  with the morphism  $d^1 : K^1 \twoheadrightarrow \text{Coker}_{\mathcal{A}}(d) \hookrightarrow K^2$  gives a  $D(\mathcal{E}, \mathcal{E}x)$ -cokernel of  $d$ . Given a morphism  $K^1 \xrightarrow{g} G$  in  $\mathcal{E}$  such that  $gd = 0$  there exists a unique  $h : \text{Coker}_{\mathcal{A}}(d) \rightarrow G$  such that  $h\pi = g$  and there exists a  $\beta : G \oplus_{\text{Coker}_{\mathcal{A}}(d)} K^2 \hookrightarrow N$  thus the following diagram commutes:

$$\begin{array}{ccccc}
 & & G & \xrightarrow{i} & N \\
 & \nearrow 0 & \uparrow g & \nearrow h & \uparrow k \\
 K^0 & \xrightarrow{d} & K^1 & \xrightarrow{d^1} & K^2 \\
 & \searrow \pi & \downarrow \alpha & \searrow 0 & \downarrow \\
 & & \text{Coker}_{\mathcal{A}}(d) & & 
 \end{array}$$

The morphism  $i : G \hookrightarrow G \oplus_{\text{Coker}_{\mathcal{A}}(d)} K^2 \hookrightarrow N$  is an inflation since it is a monomorphism in  $\mathcal{A}$  and hence, by point (4) of Definition 6.5, its cokernel is an object of  $\mathcal{E}$  thus producing the conflation  $0 \rightarrow G \xrightarrow{i} N \rightarrow \text{Coker}_{\mathcal{A}}(i) \rightarrow 0$ . On the other side let  $e : K^1 \rightarrow L$  be another  $D(\mathcal{E}, \mathcal{E}x)$ -cokernel of  $d$ ; hence there exist an inflation  $K^2 \xrightarrow{j} M$  and a morphism  $L \xrightarrow{m} M$  such that  $me = jd^1$  and (since  $ed = 0$ ) there exists a unique  $\ell : \text{Coker}_{\mathcal{A}}(d) \rightarrow L$  such that  $\ell\pi = e$ . Therefore we obtain the following commutative diagram:

$$\begin{array}{ccccc}
 & & L & \xrightarrow{m} & M \\
 & \nearrow 0 & \uparrow e & \nearrow \ell & \uparrow j \\
 K^0 & \xrightarrow{d} & K^1 & \xrightarrow{d^1} & K^2 \\
 & \searrow \pi & \downarrow \alpha & \searrow 0 & \downarrow \\
 & & \text{Coker}_{\mathcal{A}}(d) & & 
 \end{array}$$

which proves that  $\ell$  is a monomorphism (since  $m\ell\pi = j\alpha\pi$  and so  $m\ell = j\alpha$ ). This

implies that a sequence  $C^{-n+1} \xrightarrow{d_C^{-n+1}} C^{-n+2} \longrightarrow \dots \xrightarrow{d_C^{-1}} C^0$  is exact in  $\mathcal{A}$  if and only if  $C^i = D(\mathcal{E}, \mathcal{E}x)$ -cokernel of  $d_C^{i-2}$  for any  $i \geq -n-1$  and the morphism  $d_C^{-1}$  has a cokernel in  $\mathcal{E}$  and therefore point (4) of Definition 6.7 is equivalent to point

(2) of Definition 6.5. We have proved that  $(\mathcal{E}, \mathcal{E}x)$  is  $n$ -quasi-abelian. Moreover conditions (1) and (4) of Definition 6.5 imply that  $\mathcal{E}$  satisfies the hypotheses of Proposition C.11 and so  $K(\mathcal{E})/\mathcal{N}_{\mathcal{E}x} \cong D(\mathcal{A})$  and since  $D^{\geq 0}(\mathcal{A}) \cong \mathcal{RD}_{(\mathcal{E}, \mathcal{E}x)}^{\geq 0}$  we obtain that  $\mathcal{RH}(\mathcal{E}, \mathcal{E}x) \cong \mathcal{A}$  which concludes the proof.  $\square$

**Corollary 6.10.** *Let  $\mathcal{D}$  be the natural  $t$ -structure on the triangulated category  $D(\mathcal{H}_{\mathcal{D}})$  and  $i : \mathcal{E} \rightarrow \mathcal{H}_{\mathcal{D}}$  a  $n$ -tilting torsion class on  $\mathcal{H}_{\mathcal{D}}$ . Hence  $\mathcal{T}^{\leq 0} := \mathcal{D}^{\leq -n} \star \mathcal{E} \star \mathcal{E}[1] \star \dots \star \mathcal{E}[n-1]$  is an aisle in  $D(\mathcal{H}_{\mathcal{D}})$  such that  $\mathcal{E} = \mathcal{H}_{\mathcal{D}} \cap \mathcal{H}_{\mathcal{T}}$  and the pair  $(\mathcal{D}, \mathcal{T})$  is a  $n$ -tilting pair of  $t$ -structures. We will say that the  $t$ -structure  $\mathcal{T}$  is obtained by tilting  $\mathcal{D}$  with respect to the  $n$ -tilting torsion class  $\mathcal{E}$ .*

*Proof.* By Theorem 6.9 the  $n$ -tilting torsion class  $\mathcal{E}$  is a  $n$ -quasi-abelian category hence  $(\mathcal{RD}_{(\mathcal{E}, \mathcal{E}x)}, \mathcal{LD}_{(\mathcal{E}, \mathcal{E}x)})$  is a  $n$ -tilting pair of  $t$ -structures on  $D(\mathcal{E}, \mathcal{E}x) \cong D(\mathcal{H}_{\mathcal{D}})$ . The right  $t$ -structure coincides with the natural one on  $D(\mathcal{H}_{\mathcal{D}})$  (i.e.;  $\mathcal{RD}_{(\mathcal{E}, \mathcal{E}x)} = \mathcal{D}$ ) while the left  $t$ -structure satisfies  $\mathcal{LD}_{(\mathcal{E}, \mathcal{E}x)}^{\leq 0} \subseteq \mathcal{T}^{\leq 0}$  since any complex  $X^\bullet \in \mathcal{LD}_{(\mathcal{E}, \mathcal{E}x)}^{\leq 0}$  can be represented by a complex in  $K^{\leq 0}(\mathcal{E})$  and the  $\tau_{\mathcal{R}}^{\geq -n+1}(X^\bullet)$  can be represented by a complex in  $K^{[-n, 0]}(\mathcal{E})$ . On the other side since  $\mathcal{D}^{\leq -n} \cong \mathcal{RD}_{(\mathcal{E}, \mathcal{E}x)}^{\leq -n} \subseteq \mathcal{LD}_{(\mathcal{E}, \mathcal{E}x)}^{\leq 0}$  and  $\mathcal{E}[i] \subseteq \mathcal{LD}_{(\mathcal{E}, \mathcal{E}x)}^{\leq 0}$  for any  $1 \leq i \leq n-1$  we deduce that  $\mathcal{T}^{\leq 0} \subseteq \mathcal{LD}_{(\mathcal{E}, \mathcal{E}x)}^{\leq 0}$  which proves that  $\mathcal{T}^{\leq 0}$  is an aisle,  $\mathcal{E} = \mathcal{H}_{\mathcal{D}} \cap \mathcal{H}_{\mathcal{T}}$  and  $(\mathcal{D}, \mathcal{T}) = (\mathcal{RD}_{(\mathcal{E}, \mathcal{E}x)}, \mathcal{LD}_{(\mathcal{E}, \mathcal{E}x)})$  is a  $n$ -tilting pair of  $t$ -structures on  $D(\mathcal{H}_{\mathcal{D}})$ .  $\square$

Now we are able to prove the  $n$ -version of Theorem 3.6:

**Theorem 6.11.** *Let  $(\mathcal{E}, \mathcal{E}x)$  be a  $n$ -quasi-abelian category. Hence:*

$$\mathcal{LH}(\mathcal{E}, \mathcal{E}x) \cong \frac{\text{fp-}\mathcal{E}}{\text{eff-}\mathcal{E}x} \quad \mathcal{RH}(\mathcal{E}, \mathcal{E}x) \cong \left( \frac{\mathcal{E}\text{-fp}}{\mathcal{E}\text{-eff}_{\mathcal{E}x}} \right)^\circ.$$

When  $\mathcal{A}$  is an abelian category endowed with its maximal Quillen exact structure  $(\mathcal{A}, \mathcal{A}x_{\max})$  the previous equivalences give the Auslander formulas:

$$\mathcal{A} \cong \frac{\text{coh-}\mathcal{A}}{\text{eff-}\mathcal{A}} \quad \mathcal{A} \cong \left( \frac{\mathcal{A}\text{-coh}}{\mathcal{A}\text{-eff}} \right)^\circ$$

*Proof.* The second statement is dual to the first one. By the universal property of the Freyd category  $\text{fp-}\mathcal{E}$  there exists a unique functor  $L$  cokernel preserving such that the diagram below commutes:

$$\begin{array}{ccc} & \mathcal{E} & \\ \swarrow & & \searrow \\ \text{fp-}\mathcal{E} & \xrightarrow{L} & \mathcal{LH}(\mathcal{E}, \mathcal{E}x). \end{array}$$

So if  $F = \text{Coker}_{\text{fp-}\mathcal{E}}(f)$  we have  $L(F) = \text{Coker}_{\mathcal{LH}(\mathcal{E}, \mathcal{E}x)}(f)$ . The functor  $L$  is essentially surjective since any any object  $L \in \mathcal{LH}(\mathcal{E}, \mathcal{E}x)$  admits a resolution  $0 \rightarrow K^{-n} \xrightarrow{d_K^{-n}} \dots \xrightarrow{d_K^{-1}} K^0 \rightarrow L \rightarrow 0$  (due to the fact that  $\mathcal{E}$  is a  $n$ -cotilting torsion-free class in  $\mathcal{LH}(\mathcal{E}, \mathcal{E}x)$ ) thus  $L = \text{Coker}_{\mathcal{LH}(\mathcal{E}, \mathcal{E}x)}(d_K^{-1})$  and  $L \cong [K^{-n} \xrightarrow{d_K^{-n}} \dots \xrightarrow{d_K^{-1}} K^0] =: C(d_K^{-n}, \dots, d_K^{-1})$  in  $D(\mathcal{E}, \mathcal{E}x)$ .

Let us prove that  $L$  is a fully faithful functor. We notice that  $K = \text{Coker}_{\text{fp-}\mathcal{E}}(w)$  satisfies  $L(K) = 0$  if and only if  $w$  is a deflation in  $\mathcal{E}$  and so  $K \in \text{eff-}\mathcal{E}x$ . This implies that  $L$  is faith. Let us prove that  $L$  is full. Morphisms between two objects  $C(d_K^{-n}, \dots, d_K^{-1})$  and  $C(D_L^{-n}, \dots, D_L^{-1})$  in the heart are morphisms in  $D(\mathcal{E}, \mathcal{E}x)$ ,

hence there exists a complex  $C(e_M^{-n}, \dots, e_M^{-1}) \in \mathcal{LH}(\mathcal{E})$  (which we can suppose to be in  $\mathcal{LH}(\mathcal{E}, \mathcal{E}x)$  by taking its  $H_{\mathcal{E}}^0$ ) and morphisms:

$$\varphi : C(e_M^{-n}, \dots, e_M^{-1}) \rightarrow C(d_K^{-n}, \dots, d_K^{-1}); f : C(e_M^{-n}, \dots, e_M^{-1}) \rightarrow C(D_L^{-n}, \dots, D_L^{-1})$$

such that the mapping cone  $M(\varphi) \in \mathcal{N}_{\mathcal{E}x}$ . Since  $M(\varphi) \in \mathcal{N}_{\mathcal{E}x} \cap K^{\leq 0}(\mathcal{E})$  we have that its  $-1$  differential has to be a deflation:  $K^{-1} \oplus M^0 \xrightarrow{(d_K^{-1}, \varphi^0)} K^0$ . Moreover since the sequence  $K^{-2} \oplus M^{-1} \rightarrow K^{-1} \oplus M^0 \rightarrow K^0$  is exact we deduce that  $\text{Coker}_{\text{fp-}\mathcal{E}}(e_M^{-1}) \hookrightarrow \text{Coker}_{\text{fp-}\mathcal{E}}(d_K^{-1})$  thus it lifts to a morphism in  $\frac{\text{fp-}\mathcal{E}}{\text{eff-}\mathcal{E}x}$  between  $\text{Coker}_{\text{fp-}\mathcal{E}}(d_K^{-1})$  and  $\text{Coker}_{\text{fp-}\mathcal{E}}(D_L^{-1})$ .  $\square$

**Corollary 6.12.** *Let  $(\mathcal{E}, \mathcal{E}x)$  be a  $n$ -quasi-abelian category and  $\overline{\mathcal{E}x}$  a Quillen exact structure on  $\mathcal{E}$  finer than  $\mathcal{E}x$ . Hence the class*

$$\overline{\text{eff-}\mathcal{E}x} := \{\text{Coker}_{\mathcal{LH}(\mathcal{E}, \mathcal{E}x)}(w) \mid w \text{ is a deflation in } \overline{\mathcal{E}x}\}$$

*is a Serre subcategory of  $\mathcal{LH}(\mathcal{E}, \mathcal{E}x)$  and  $\mathcal{LH}(\mathcal{E}, \overline{\mathcal{E}x}) \cong \frac{\mathcal{LH}(\mathcal{E}, \mathcal{E}x)}{\overline{\text{eff-}\mathcal{E}x}}$ .*

**Corollary 6.13.** *Let  $\mathcal{E}$  be a 1-quasi-abelian category. Hence*

$$\mathcal{LH}(\mathcal{E}) \cong \frac{\text{coh-}\mathcal{E}}{\text{eff-}\mathcal{E}} \quad \mathcal{RH}(\mathcal{E}) \cong \left( \frac{\mathcal{E}\text{-coh}}{\mathcal{E}\text{-eff}} \right)^{\circ}.$$

**Remark 6.14.** Let consider  $(\mathcal{E}, \mathcal{E}x)$  a  $n$ -quasi-abelian category which is not a  $n-1$ -quasi-abelian category (i.e.; such that  $\mathcal{RD}_{(\mathcal{E}, \mathcal{E}x)}^{\leq -n} \subseteq \mathcal{LD}_{(\mathcal{E}, \mathcal{E}x)}^{\leq 0}$  but  $\mathcal{RD}_{(\mathcal{E}, \mathcal{E}x)}^{\leq -n+1} \not\subseteq \mathcal{LD}_{(\mathcal{E}, \mathcal{E}x)}^{\leq 0}$ ) with  $n \geq 3$ . Hence for any Quillen exact structure  $\overline{\mathcal{E}x}$  on  $\mathcal{E}$  finer than  $\mathcal{E}x$  (i.e.; which contains the conflations of  $\mathcal{E}x$ ) we have that  $(\mathcal{E}, \overline{\mathcal{E}x})$  is a  $n$ -quasi-abelian category which is not a  $n-1$ -quasi-abelian category. If it was true that  $\mathcal{RD}_{(\mathcal{E}, \overline{\mathcal{E}x})}^{\leq -n+1} \subseteq \mathcal{LD}_{(\mathcal{E}, \overline{\mathcal{E}x})}^{\leq 0}$  hence any object  $L \in \mathcal{LH}(\mathcal{E}, \mathcal{E}x)$  which has a presentation  $0 \rightarrow K^{-n} \xrightarrow{d_K^{-n}} \dots \xrightarrow{d_K^{-1}} K^0 \rightarrow L \rightarrow 0$  would short in  $\mathcal{LH}(\mathcal{E}, \overline{\mathcal{E}x})$  i.e.;  $d_K^{-n+2}$  would have a kernel (computed in  $\mathcal{LH}(\mathcal{E}, \overline{\mathcal{E}x})$ ) which belongs to  $\mathcal{E}$  but (since  $\mathcal{E}$  is fully faithful in  $\mathcal{LH}(\mathcal{E}, \mathcal{E}x)$ ) this would be a kernel for  $d_K^{-n+2}$  also in  $\mathcal{LH}(\mathcal{E}, \mathcal{E}x)$  which contradicts the hypothesis.

We are now able to prove the  $n$  version of Theorem 1.14.

An object in  $\{n\text{-quasi-abelian categories}\}$  is a  $n$ -quasi-abelian category  $(\mathcal{E}, \mathcal{E}x)$ .

**Theorem 6.15.** *There is a one to one correspondence between the classes*

$$\begin{array}{ccc} \{n\text{-quasi-abelian categories}\} & \longleftrightarrow & \{n\text{-tilting pairs of } t\text{-structures}\} \\ \uparrow & & \uparrow \\ \mathcal{E} & \xleftrightarrow{\mathcal{RH}(\mathcal{E}, \mathcal{E}x) \cap \mathcal{LH}(\mathcal{E}, \mathcal{E}x)} & (\mathcal{RD}_{(\mathcal{E}, \mathcal{E}x)}, \mathcal{LD}_{(\mathcal{E}, \mathcal{E}x)}) \text{ on } \mathcal{C} = D(\mathcal{E}, \mathcal{E}x) \\ \downarrow & & \downarrow \\ \{n\text{-tilting torsion classes}\} & \longleftrightarrow & \{n\text{-cotilting torsion-free classes}\} \\ \downarrow & & \downarrow \\ \mathcal{E} \text{ in } \mathcal{RH}(\mathcal{E}, \mathcal{E}x) & \longleftrightarrow & \mathcal{E} \text{ in } \mathcal{LH}(\mathcal{E}, \mathcal{E}x). \end{array}$$

*Proof.* Let  $(\mathcal{E}, \mathcal{E}x)$  be a  $n$ -quasi-abelian category. By Lemma 6.4  $(\mathcal{RD}_{(\mathcal{E}, \mathcal{E}x)}, \mathcal{LD}_{(\mathcal{E}, \mathcal{E}x)})$  is a  $n$ -tilting pair of  $t$ -structures on  $D(\mathcal{E}, \mathcal{E}x)$  and by Remark 6.8  $\mathcal{E}$  is a  $n$ -tilting torsion class in  $\mathcal{RH}(\mathcal{E}, \mathcal{E}x)$  (respectively  $\mathcal{E}$  is a  $n$ -tilting torsion-free class in  $\mathcal{LH}(\mathcal{E}, \mathcal{E}x)$ ).

If  $(\mathcal{D}, \mathcal{T})$  is a  $n$ -tilting pair of  $t$ -structures in  $\mathcal{C}$  hence by Remark 6.6  $\mathcal{E} = \mathcal{H}_{\mathcal{D}} \cap \mathcal{H}_{\mathcal{T}}$  is a  $n$ -quasi-abelian category and by Proposition 2.5  $(\mathcal{D}, \mathcal{T}) = (\mathcal{RD}_{(\mathcal{E}, \mathcal{E}x)}, \mathcal{LD}_{(\mathcal{E}, \mathcal{E}x)})$ .



By Theorem 6.9 if  $\mathcal{E}$  is a  $n$ -tilting torsion class in  $\mathcal{A}$  hence  $\mathcal{E}$  is  $n$ -quasi-abelian and  $\mathcal{A} \cong \mathcal{RH}(\mathcal{E}, \mathcal{E}x)$  which concludes the proof.  $\square$

## APPENDIX A. MAXIMAL QUILLEN EXACT STRUCTURE

Let us briefly recall the notion of Quillen exact structure on an additive category  $\mathcal{E}$  and some recent results on the maximal Quillen exact structure on  $\mathcal{E}$ . We refer to [Kel90] and [Büh10].

**A.1.** An *exact category* is the data  $(\mathcal{E}, \mathcal{E}x)$  of an additive category  $\mathcal{E}$  and a class  $\mathcal{E}x$  of exact sequences  $A \xrightarrow{i} B \xrightarrow{p} C$  called *conflations* ( $i$ , called *inflation*, is a kernel of  $p$  while  $p$ , called a *deflation*, is a cokernel of  $i$ ) satisfying the following axioms:

- Ex0: The identity morphism of the zero object is a deflation.
- Ex1: The composition of two deflations is a deflation.
- Ex1°: The composition of two inflations is an inflation.
- Ex2: The push-out of an inflation along an arbitrary morphism exists and yields an inflation.
- Ex2°: The pull-back of a deflation along an arbitrary morphism exists and yields a deflation.

We call  $\mathcal{E}x$  an *exact structure* on  $\mathcal{E}$ .

In general an additive category  $\mathcal{E}$  can admit different exact structures. In particular the previous axioms imply that any split short exact sequence is a conflation for any exact structure on  $\mathcal{E}$ . Hence any additive category  $\mathcal{E}$  admits a *minimal exact structure* whose conflations are the split short exact sequences.

Recently many advances have been done also for the dual problem: does  $\mathcal{E}$  admit a *maximal exact structure*? Due to the definition of Quillen exact structure the natural candidate for being the class of conflations for the maximal exact structure on  $\mathcal{E}$  is the class of all kernel-cokernel pairs stable by push-outs and pull-backs. In [Rum11] Rump proved that any additive category admits a maximal Quillen exact structure and in [SW11] Sieg and Wegner proved that for additive categories with kernels and cokernels (or equivalently 2-quasi-abelian categories) this maximal exact structure coincides with the class of all stable kernel-cokernel pairs. Crivei generalized the result of Sieg and Wegner as follows:

**Proposition A.2.** [Cri12, Theorem 3.5] *Let  $\mathcal{E}$  be a weakly idempotent complete additive category (i.e., additive category in which every section has a cokernel, or equivalently, every retraction has a kernel). The class of all kernel-cokernel pairs stable by push-outs and pull-backs is a Quillen exact structure on  $\mathcal{E}$  and hence it is the maximal one.*

**Remark A.3.** Any projectively complete category is weakly idempotent complete and additive. In particular any 2-quasi-abelian category is projectively complete. Let us recall that if  $(\mathcal{E}, \mathcal{E}x)$  is a Quillen exact structure on a weakly idempotent complete category  $\mathcal{E}$  we have: if  $gf$  is a deflation hence  $g$  is a deflation too ([Büh10, Proposition 7.6.]).

**Definition A.4.** Let  $(\mathcal{E}, \mathcal{E}x)$  be an exact category. A complex  $X^\bullet$  with entries in  $\mathcal{E}$  is called *acyclic* if each differential  $d^n : X^n \rightarrow X^{n+1}$  decomposes in  $\mathcal{E}$  as

$d^n = m_n \circ e_n : X^n \xrightarrow{e_n} D^n \xrightarrow{m_n} X^{n+1}$  where  $m_n$  is an inflation,  $e_n$  is a deflation and the sequence  $D^n \xrightarrow{m_n} X^{n+1} \xrightarrow{e_{n+1}} D^{n+1}$  belongs to  $\mathcal{E}x$  for any  $n \in \mathbb{Z}$ .

**A.5. The Derived Category of a projectively complete exact category** [Nee90]. Let us recall (as done by Neeman in [Nee90]) how one can associate a “derived” category to a projectively complete exact category  $(\mathcal{E}, \mathcal{E}x)$ . Let consider  $K(\mathcal{E})$  the homotopy category of chain complexes in  $\mathcal{E}$  and let  $\mathcal{N}_{\mathcal{E}x}$  be the full subcategory of  $K(\mathcal{E})$  whose objects are acyclic complexes. By [Nee90, Lemma 1.1]  $\mathcal{N}_{\mathcal{E}x}$  is a triangulated subcategory (in the proof one do not need the idempotent completion hypothesis). By [Nee90, Lemma 1.2 and Remark 1.8]  $\mathcal{N}_{\mathcal{E}x}$  is a thick full triangulated subcategory of  $K(\mathcal{E})$  if and only if  $\mathcal{E}$  is projectively complete. Hence the *derived category*  $D(\mathcal{E})$  of a projectively complete exact category  $(\mathcal{E}, \mathcal{E}x)$  is by definition the quotient (as triangulated categories) of  $K(\mathcal{E})$  by  $\mathcal{N}_{\mathcal{E}x}$ . Whenever the exact structure on  $\mathcal{E}$  is not specified we will endow  $\mathcal{E}$  with its maximal Quillen exact structure which is, by Crivei result A.2, the one formed by all kernel-cokernel pairs stable by push-outs and pull-backs.

**Lemma A.6.** *Let  $(\mathcal{E}, \mathcal{E}x)$  be a projectively complete category endowed with a Quillen exact structure. For any  $X^\bullet \in K^{\leq 0}(\mathcal{E})$ ;  $Y^\bullet \in K^{\geq 0}(\mathcal{E})$  we have*

$$D(\mathcal{E}, \mathcal{E}x)(X^\bullet, Y^\bullet) = K(\mathcal{E})(X^\bullet, Y^\bullet).$$

*Proof.* Given  $\alpha \in D(\mathcal{E}, \mathcal{E}x)(X^\bullet, Y^\bullet)$  the composition  $i : X^0[0] \rightarrow X^\bullet \xrightarrow{\alpha} Y^\bullet \rightarrow Y[0]$  produces a morphism  $i : X^0 \rightarrow Y^0$  in  $\mathcal{E}$  such that  $d_{Y^\bullet}^0 \circ i = 0$  and  $i \circ d_{X^\bullet}^{-1} = 0$ . Hence  $i$  induces a morphism in  $K(\mathcal{E})$  which represents  $\alpha$ .  $\square$

## APPENDIX B. FREYD CATEGORIES AND COHERENT FUNCTORS

In the following we will consider  $\mathcal{C}$  a category in the classical terminology for which any homomorphism class  $\mathcal{C}(X, Y)$  with  $X, Y$  objects in  $\mathcal{C}$  is a set. Some authors define this a locally small category in order to underline that its homomorphism form a set. The wider notion of category which permits to consider also homomorphism which does not form a set is very convenient once working with localization procedure.

**Definition B.1.** Let us recall that a category  $\mathcal{C}$  is called:

- (1) *pre-additive* if any hom-set is a group and the composition is bilinear;
- (2) *additive* if it is pre-additive with zero object and biproducts;
- (3) *idempotent complete* <sup>2</sup> if any idempotent splits;
- (4) *projectively complete* <sup>3</sup> when it is additive and idempotent complete.

In the following we will use the following notation:  $\mathcal{C}$  for a pre-additive category,  $\mathcal{P}$  for a projectively complete category,  $\mathcal{E}$  for a Quillen exact category (see A.1) and  $\mathcal{A}$  for an abelian category.

**B.2.** Let us start considering  $\mathcal{C}$  a pre-additive category, we will denote by  $\mathcal{C}(A, B)$  the group of morphisms between  $A$  and  $B$  and by  $\mathcal{C}^\circ$  its opposite category (hence  $\mathcal{C}^\circ(B, A) = \mathcal{C}(A, B)$ ). In what follows, any full subcategory of  $\mathcal{C}$  will be strictly full (i.e., closed under isomorphisms). Any functor between pre-additive categories will

<sup>2</sup> It also called Karoubian by some authors.

<sup>3</sup> It is also called Cauchy complete in [Str95], or amenable by [Fre66].

be an additive functor (i.e.,  $F : \mathcal{C} \rightarrow \mathcal{C}'$  such that for any  $X, Y \in \mathcal{C}$  the morphism  $\mathcal{C}(X, Y) \rightarrow \mathcal{C}'(F(X), F(Y))$  is an abelian group homomorphism).

The main interest in taking in account the generality of pre-additive categories has been firstly remarked by Mitchell in [Mit72] since any ring (associative with unit)  $R$  can be regarded as a pre-additive category with precisely one object  $*$  such that its endomorphism group is  $R(*, *) = R$ . In the following when we will write  $\mathcal{C} = R$  we will mean exactly this example. Let us note that also the zero ring  $0$  (with  $0 = 1$ ) is a pre-additive category with a single object. Moreover ring morphisms correspond exactly to additive functors. Hence we denote by  $\mathcal{R}ing$  the category of rings (associative with unit) which is the full subcategory of pre-additive categories with exactly one object.

Following the notation of [Kra15] [AK02] and [Str95], inspired by Mitchell work [Mit72], we denote by  $\mathbf{Mod}\text{-}\mathcal{C}$  the enriched category of additive contravariant functors (i.e.  $F : \mathcal{C}^\circ \rightarrow \mathcal{A}b$ ) from  $\mathcal{C}$  to the category  $\mathcal{A}b$  of abelian groups and by  $\mathcal{C}\text{-}\mathbf{Mod}$  the one of covariant functors (i.e.  $G : \mathcal{C} \rightarrow \mathcal{A}b$ ). Hence  $\mathbf{Mod}\text{-}\mathcal{C}^\circ$  is isomorphic to the category  $\mathcal{C}\text{-}\mathbf{Mod}$ . The following functors are the enriched version of the ones firstly studied by Yoneda:

$$\begin{array}{ccc} Y_{\mathcal{C}} : \mathcal{C} & \longrightarrow & \mathbf{Mod}\text{-}\mathcal{C} \\ X & \longmapsto & \mathcal{C}_X := \mathcal{C}(\_, X) \end{array} \quad \begin{array}{ccc} {}_{\mathcal{C}}Y : \mathcal{C} & \longrightarrow & (\mathcal{C}\text{-}\mathbf{Mod})^\circ \\ X & \longmapsto & {}_X\mathcal{C} := \mathcal{C}(X, \_) \end{array}$$

they admit an additive analogue of the Yoneda Lemma.

Let recall some results on these categories of functors:

**Proposition B.3.** ([ML98, III.§2, §7]) *Let  $\mathcal{C}$  be a pre-additive category. The followings hold true:*

(i): **Yoneda Lemma:** *let  $X$  be an object in  $\mathcal{C}$  and  $M$  in  $\mathbf{Mod}\text{-}\mathcal{C}$  (respectively  $N$  in  $\mathcal{C}\text{-}\mathbf{Mod}$ ). Then*

$$\mathbf{Mod}\text{-}\mathcal{C}(\mathcal{C}_X, M) \cong M(X) \quad \text{and} \quad \mathcal{C}\text{-}\mathbf{Mod}({}_X\mathcal{C}, N) \cong N(X)$$

*and hence the functors  ${}_{\mathcal{C}}Y$  and  $Y_{\mathcal{C}}$  are fully faithful. A functor  $M \cong \mathcal{C}_X \in \mathbf{Mod}\text{-}\mathcal{C}$  (respectively  $N \cong {}_X\mathcal{C} \in \mathcal{C}\text{-}\mathbf{Mod}$ ) with  $X \in \mathcal{C}$  is called a representable functor and hence the essential image of  $Y_{\mathcal{C}}$  (respectively  ${}_{\mathcal{C}}Y$ ) is provided by the full subcategory of representable functors and it is denoted by  $Y_{\mathcal{C}}(\mathcal{C})$  (respectively  ${}_{\mathcal{C}}Y(\mathcal{C})$ ).*

*Let  $\mathcal{C}$  be a small pre-additive category:*

(ii): *The category  $\mathbf{Mod}\text{-}\mathcal{C}$  (respectively  $\mathcal{C}\text{-}\mathbf{Mod}$ ) is an abelian complete and cocomplete category (i.e. its small inductive and projective limits are representable) whose filtered inductive limits are exact. Moreover representable functors generate  $\mathbf{Mod}\text{-}\mathcal{C}$  since if  $\mathbf{Mod}\text{-}\mathcal{C}(\mathcal{C}_X, F) \cong F(X) = 0$  for any  $X \in \mathcal{C}$  hence  $F = 0$ .*

(iii): *Any  $M \in \mathbf{Mod}\text{-}\mathcal{C}$  (respectively  $N \in \mathcal{C}\text{-}\mathbf{Mod}$ ) is an inductive limit of representable functors. Moreover if  $\mathcal{C}$  is cocomplete (respectively complete) the functor  ${}_{\mathcal{C}}Y$  (respectively  $Y_{\mathcal{C}}$ ) admits a left adjoint  $\varinjlim : \mathbf{Mod}\text{-}\mathcal{C} \rightarrow \mathcal{C}$  (respectively  $\varprojlim := (\varinjlim)^\circ : (\mathcal{C}\text{-}\mathbf{Mod})^\circ \rightarrow \mathcal{C}$ ).*

(iv): *Any representable functor  $\mathcal{C}_X$  is projective and compact in  $\mathbf{Mod}\text{-}\mathcal{C}$  and hence the functor  $\mathbf{Mod}\text{-}\mathcal{C}(\mathcal{C}_X, \_)$  commutes with all colimits; moreover  $P \in \mathbf{Mod}\text{-}\mathcal{C}$  is projective and compact if and only if it is a direct summand of a finite direct sum of representable functors. (Respectively: any representable functor  ${}_X\mathcal{C}$  is projective and compact in  $\mathcal{C}\text{-}\mathbf{Mod}$  and hence the*

*functor  $\mathcal{C}\text{-Mod}({}_X\mathcal{C}, \_)$  commutes with all colimits, moreover  $Q \in \mathcal{C}\text{-Mod}$  is projective and compact if and only if it is a direct summand of a finite direct sum of representable functors).*

**Remark B.4.** Let  $\mathcal{C}$  be a pre-additive category (not necessarily small), one can perform a *projective completion* of  $\mathcal{C}$  formally adding the zero objects and finitely direct sums of objects in  $\mathcal{C}$  and hence taking its idempotent completion. We denote by  $\text{add}(\mathcal{C})$  the projective completion of  $\mathcal{C}$  (see for example [BS01]). Let us denote by  $\text{proj-}\mathcal{C}$  (respectively  $\mathcal{C}\text{-proj}$ ) the full subcategory of  $\text{Mod-}\mathcal{C}$  (respectively of  $\mathcal{C}\text{-Mod}$ ) whose objects are direct summands of finite direct sums of representable functors (we note that natural transformations between two such objects always form a set). Hence  $\text{add}(\mathcal{C})$ ,  $\text{proj-}\mathcal{C}$  and  $\mathcal{C}\text{-proj}$  are equivalent. Whenever  $\mathcal{P}$  is a projectively complete category (see Definition B.1) the category  $\text{proj-}\mathcal{P}$  coincides with the full subcategory of  $\text{Mod-}\mathcal{P}$  of representable functors. We remark that any additive functor  $F : \mathcal{C}^\circ \rightarrow \mathcal{A}b$  can uniquely be extended to an additive functor  $\overline{F} : (\text{proj-}\mathcal{C})^\circ \rightarrow \mathcal{A}b$ .

**B.5. Coherent Functors** In his paper for the Proc. Conf. Categorical Algebra (La Jolla, Calif., 1965) [Aus66] Auslander introduced the study of *coherent functors* in the category  $\text{Mod-}\mathcal{A}$  with  $\mathcal{A}$  an abelian category (a “genetic” introduction to this theme can be found in [Har98]). In the same collection Freyd [Fre66] introduced the study of the *Freyd category of finitely presented functors* associated to a projectively complete category  $\mathcal{P}$ . These theories, and hence the related vocabulary, are widely inspired by the theory of finitely presented and coherent modules over a ring  $R$  which is also the easiest case (pre-additive category with a single object see B.2).

The basic idea is that whatever one knows on finitely presented, respectively coherent, modules over a ring has its counterpart for finitely presented, respectively coherent, functors in  $\text{Mod-}\mathcal{C}$  (see Appendix B). It is well known ([Bou89, Ch.I], [Bos13, §1.5]) that, given a ring  $R$ , right coherent modules  $\text{coh-}R$  form a full abelian subcategory of all right  $R$  modules  $\text{Mod-}R$  while finitely presented modules  $\text{fp-}R$  form a full projectively complete subcategory of  $\text{Mod-}R$  admitting cokernels. The category  $\text{fp-}R$  is an abelian subcategory of  $\text{Mod-}R$  if and only if the ring  $R$  is coherent and in that case coherent and finitely presented modules coincide:  $\text{coh-}R = \text{fp-}R$  (these theorems go back to Henri Cartan); while in general finitely generated modules form a full projectively complete subcategory of  $\text{Mod-}R$  and  $\text{fg-}R$  is an abelian subcategory of  $\text{Mod-}R$  if and only if the ring  $R$  is right noetherian and in this case coherent modules coincide with the finitely generated ones:  $\text{coh-}R = \text{fp-}R = \text{fg-}R$ . The proofs of these statements are based on the fact that  $R^n$  is a projective compact object in  $\text{Mod-}R$  and hence the functor  $\text{Hom}_R(R^n, \_)$  commutes with all colimits (and limits too). In Proposition B.3 point (iv) we stated that any direct summand of a finite direct sum of representable functors in  $\text{Mod-}\mathcal{C}$  (i.e., an element in  $\text{proj-}\mathcal{C}$ ) is projective and compact hence Cartan results extend to coherent functors replacing the role of  $R^n$  by that of a direct summand of a representable functor  $\mathcal{C}_X$  (hence an object in  $\text{proj-}\mathcal{C}$ ). Let us recall that, by Remark B.4, the categories  $\text{Mod-}\mathcal{C}$  and  $\text{Mod-proj}(\mathcal{C})$  are equivalent, hence from now on given any pre-additive category we will pass to its projective completion  $\mathcal{P} := \text{proj}(\mathcal{C})$ .

Let us briefly summarize the main results on this subject whose main references are: Freyd [Fre66], Auslander [Aus66] and Beligiannis [Bel00]. In particular Freyd work has been further investigated and developed by Beligiannis in his very inspiring paper [Bel00] to which we widely refer to. Let  $\mathcal{P}$  be a projectively complete category.

**Definition B.6.** An object  $F \in \text{Mod-}\mathcal{P}$  is called *finitely generated* if it is generated by a representable functor: i.e. there exists an epimorphism  $\mathcal{P}_X \twoheadrightarrow F$  with  $X \in \mathcal{P}$ . An object  $F \in \text{Mod-}\mathcal{P}$  is called *finitely presented* if it fits into an exact sequence in  $\text{Mod-}\mathcal{P}$ :

$$\mathcal{P}_{X_1} \longrightarrow \mathcal{P}_{X_2} \longrightarrow F \longrightarrow 0$$

with  $X_i \in \mathcal{P}$  for  $i = 1, 2$ . An object  $F$  finitely generated is called *coherent* if any subobject  $G \hookrightarrow F$  finitely generated is finitely presented too. Hence any finitely generated subfunctor of a coherent functor is coherent too. We will denote by **fg- $\mathcal{P}$** , respectively **fp- $\mathcal{P}$** , respectively **coh- $\mathcal{P}$**  the full subcategory of  $\text{Mod-}\mathcal{P}$  whose objects are the finitely generated, respectively finitely presented, respectively coherent functors.

We obtain the following commutative diagram of fully faithful functors:

$$(4) \quad \begin{array}{ccccc} & \mathcal{P} & & & \\ & \downarrow P_{\mathcal{P}} & \searrow Y_{\mathcal{P}} & & \\ \text{coh-}\mathcal{P}^c & \longrightarrow & \text{fp-}\mathcal{P}^c & \longrightarrow & \text{fg-}\mathcal{P}^c \longrightarrow \text{Mod-}\mathcal{P} \end{array}$$

where by definition  $P_{\mathcal{P}}$  is the Yoneda functor whose codomain is restricted to finitely presented functors. We remark that the class of natural transformations between finitely generated functors is a set since if  $\mathcal{P}_X \twoheadrightarrow F$  and  $\mathcal{P}_Y \twoheadrightarrow G$  any morphism  $\alpha : F \rightarrow G$  can be lifted to a morphism from  $\mathcal{P}_X \rightarrow \mathcal{P}_Y$  which by the Yoneda Lemma is an element of the group  $\mathcal{P}(X, Y)$ .

Following Beligiannis [Bel00, Definition 3.1] the categories  $\text{fp-}\mathcal{P}$  and  $(\mathcal{P}\text{-fp})^\circ$  are called the *Freyd categories* of  $\mathcal{P}$ . In [Bel00] Beligiannis used the notation:  $\mathcal{A}(\mathcal{P}) = \text{fp-}\mathcal{P}$  while  $\mathcal{B}(\mathcal{P}) = (\mathcal{P}\text{-fp})^\circ$  which would be useful in the sequel when we will need to perform the double construction  $\mathcal{A}(\mathcal{B}(\mathcal{P}))$ .

**B.7.** Given  $\mathcal{P}$  a projectively complete category, Freyd proved in [Fre66] that  $\text{fp-}\mathcal{P}$  is projectively closed with cokernels and that an object  $F$  is projective in  $\text{fp-}\mathcal{P}$  (i.e., for any epimorphism  $p : G_1 \twoheadrightarrow G_2$  in  $\text{fp-}\mathcal{P}$  the map  $\text{fp-}\mathcal{P}(F, G_1) \rightarrow \text{fp-}\mathcal{P}(F, G_2)$  is surjective) if and only if  $F \cong \mathcal{P}_X$  (see B.3.(i)) since any finitely presented functor  $F$  is finitely generated and so there exists  $\mathcal{P}_Y \twoheadrightarrow F$  and hence, by the projectivity of  $F$ , it splits so  $F$  is representable since  $\mathcal{P}$  is projectively complete.

Let us recall the definition of generating family:

**Definition B.8.** Let  $\mathcal{C}$  be a pre-additive category. A family of objects  $\mathcal{G}$  is called a *generating family* if for any non zero morphism  $f : C \rightarrow D$  in  $\mathcal{C}$  there exists a morphism  $h : G \rightarrow C$  with  $G$  in  $\mathcal{G}$  such that  $f \circ h \neq 0$ . A co-generating family of  $\mathcal{C}$  is a generator family of  $\mathcal{C}^\circ$ .

**Remark B.9.** Let  $\mathcal{P}$  be a projectively complete category. Objects in  $\mathcal{P}$  form a generating (respectively co-generating) family of projective (respectively injective) objects for  $\mathcal{A}(\mathcal{P})$  (respectively  $\mathcal{B}(\mathcal{P})$ ).

**B.10.** In [Bel00] Beligiannis, following Freyd, proved that the pair  $(\text{fp-}\mathcal{P}, P_{\mathcal{P}})$  is “universal” between the projectively closed categories with cokernels “containing an image” of  $\mathcal{P}$ : (given any other projectively closed category  $\mathcal{D}$  with cokernels and an additive functor  $F : \mathcal{P} \rightarrow \mathcal{D}$  there exists unique a functor  $F^c : \text{fp-}\mathcal{P} \rightarrow \mathcal{D}$  cokernel preserving such that  $F^c \circ P_{\mathcal{P}} = F$ . See Beligiannis work for a translation of this universality property in terms of an adjunction (our  $F^c$  is  $F^!$ ). In [Bel00,

Lemma 3.3] the author proved that the functor  $P_{\mathcal{P}}$  always preserves kernels and moreover it admits a left adjoint  $\Phi_{\mathcal{P}}$  if and only if  $\mathcal{P}$  has cokernels.

**Definition B.11.** (Freyd [Fre66, page 103], Beligiannis [Bel00, §4]). A projectively complete category  $\mathcal{P}$  is called *right coherent* (respectively *left coherent*) if for any  $X \in \mathcal{P}$  the functor  $\mathcal{P}_X$  (respectively  ${}_X\mathcal{P}$ ) is coherent.  $\mathcal{P}$  is called *coherent*<sup>4</sup> if it is both left and right coherent. A pre-additive category  $\mathcal{C}$  is called (respectively right, respectively left) coherent if and only if the category  $\text{proj}(\mathcal{C})$  is (respectively right, respectively left) coherent.

**Remark B.12.** In the case of an additive category Definition B.6 says that  $F \in \text{Mod-}\mathcal{P}$  is finitely generated if there exists an epimorphism  $P \twoheadrightarrow F$  with  $P \in \text{proj-}\mathcal{P}$  but, since for any object  $P \in \text{proj-}\mathcal{P}$  there exist a  $Q \in \text{proj-}\mathcal{P}$  and  $X \in \mathcal{P}$  such that  $P \oplus Q \cong \mathcal{P}_X$ , any epimorphism  $P \twoheadrightarrow F$  can be extended to  $\mathcal{P}_X \twoheadrightarrow F$  and so a functor  $F$  is coherent in an additive category  $\mathcal{P}$  if and only if it fits into an exact sequence in  $\text{Mod-}\mathcal{P}$ :  $\mathcal{P}_{X_1} \longrightarrow \mathcal{P}_{X_2} \longrightarrow F \longrightarrow 0$  with  $X_i \in \mathcal{P}$  for  $i = 1, 2$ . In the case of a ring  $R$  regarded as a pre-additive category the category of functors  $\text{Mod-}R$  coincides with that of right  $R$ -modules and  $\text{proj-}R$  is the category of finitely generated projective modules while there is only a representable functor in  $\text{Mod-}R$  which is the ring  $R$ . In this case Definition B.6 provides the usual definitions of finitely generated, finitely presented and coherent modules in  $\text{Mod-}R$  and so a ring  $R$  is (respectively right, respectively left) coherent if and only if it is (respectively right, respectively left) coherent as a pre-additive category following Definition B.11.

We propose in Proposition B.15 a proof of the fact that  $\text{coh-}\mathcal{E}$  is an abelian category for any  $\mathcal{E}$  projectively complete category. This statement, which is probably originally due to Henri Cartan, is proposed in its version for a ring  $R$ , as an exercise in Bourbaki [Bou89, §2 Exercise 11] and explained in great detail in [Bos13, §1.5]. This is a translation of Bosch proofs in the language of pre-additive categories.

**Proposition B.13.** *Let*

$$0 \longrightarrow F_1 \xrightarrow{i} F \xrightarrow{j} F_2 \longrightarrow 0$$

*be a short exact sequence in  $\text{Mod-}\mathcal{E}$ . Hence:*

- (1) *if  $F$  is finitely generated then  $F_2$  is finitely generated;*
- (2) *if  $F_1, F_2$  are finitely generated then  $F$  is finitely generated;*
- (3) *if  $F_1, F_2$  are finitely presented then  $F$  is finitely presented too.*

*Proof.* (1) is clear since if  $P \xrightarrow{p} F$  with  $P \in \text{proj-}\mathcal{E}$  hence  $jp : P \twoheadrightarrow F_2$  proves that  $F_2$  is finitely generated.

(2) If  $p_i : P_i \twoheadrightarrow F_i$  with  $P_i \in \text{proj-}\mathcal{E}$  and  $i = 1, 2$  let consider the following diagram with exact rows:

$$\begin{array}{ccccccc} 0 & \longrightarrow & P_1 & \longrightarrow & P_1 \oplus P_2 & \longrightarrow & P_2 \longrightarrow 0 \\ & & p_1 \downarrow & & (p_1, \tilde{p}_2) \downarrow & & \downarrow p_2 \\ 0 & \longrightarrow & F_1 & \xrightarrow{i} & F & \xrightarrow{j} & F_2 \longrightarrow 0 \end{array}$$

<sup>4</sup>We remark that the notion of coherent additive category has nothing to do with the one proposed by Peter Johnstone for a general category.

The map  $\tilde{p}_2$  exists since  $j$  is an epimorphism and  $P_2$  is projective and hence  $(p_1, \tilde{p}_2)$  is an epimorphism too which proves that  $F$  is finitely generated.

(3) Let  $Q_i \rightarrow P_i \rightarrow F_i \rightarrow 0$  a presentation of  $F_i$  with  $P_i, Q_i \in \text{proj-}\mathcal{E}$  for any  $i = 1, 2$ . Hence the commutative diagram with exact rows and columns:

$$\begin{array}{ccccccc}
& & 0 & & 0 & & 0 \\
& & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & \text{Ker}(p_1) & \longrightarrow & \text{Ker}(p_1, \tilde{p}_2) & \longrightarrow & \text{Ker}(p_2) \longrightarrow 0 \\
& & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & P_1 & \longrightarrow & P_1 \oplus P_2 & \longrightarrow & P_2 \longrightarrow 0 \\
& & \downarrow p_1 & & \downarrow (p_1, \tilde{p}_2) & & \downarrow p_2 \\
0 & \longrightarrow & F_1 & \xrightarrow{i} & F & \xrightarrow{j} & F_2 \longrightarrow 0 \\
& & \downarrow & & \downarrow & & \downarrow \\
& & 0 & & 0 & & 0
\end{array}$$

proves that  $\text{Ker}(p_1, \tilde{p}_2)$  is finitely presented by applying the previous point (2) since  $\text{Ker}(p_1)$  and  $\text{Ker}(p_2)$  are finitely generated (by  $Q_1$  and  $Q_2$  respectively).  $\square$

**Proposition B.14.** *The following conditions are equivalent:*

- (1): *the functor  $F$  is finitely presented;*
- (2):  *$F$  is finitely generated and for any epimorphism  $\psi : G \twoheadrightarrow F$  with  $G$  finitely generated  $\text{Ker}(\psi)$  is finitely generated.*

From which we deduce that the following conditions are equivalent:

- (i): *the functor  $F$  is coherent;*
- (ii):  *$F$  is finitely generated and for any morphism  $\phi : G \rightarrow F$  with  $G$  finitely generated  $\text{Ker}(\phi)$  is finitely generated too.*

*Proof.* Let us prove that (1) is equivalent to (2). It is clear that (2) implies (1).

On the other side let  $Q \xrightarrow{f} P \xrightarrow{g} F \rightarrow 0$  be a presentation with  $P, Q \in \text{proj-}\mathcal{E}$  and  $\psi : G \twoheadrightarrow F$  with  $G$  finitely generated. Then let consider the commutative diagram

$$\begin{array}{ccccccc}
0 & \longrightarrow & \text{Ker}(g) & \longrightarrow & P & \xrightarrow{g} & F \longrightarrow 0 \\
& & \downarrow u & & \downarrow v & & \parallel \\
0 & \longrightarrow & \text{Ker}(\psi) & \longrightarrow & G & \xrightarrow{\psi} & F \longrightarrow 0 \\
& & \downarrow & & \downarrow & & \\
& & \text{Coker}(u) & \xrightarrow{\cong} & \text{Coker}(v) & & \\
& & \downarrow & & \downarrow & & \\
& & 0 & & 0 & & 
\end{array}$$

where we use the projectivity of  $P$  in order to find the dotted arrow  $v$  while for any such  $v$  the universal property of  $\text{Ker}(\psi)$  permits to recover a unique  $u$  making the diagram commutative. Hence the short exact sequence  $0 \rightarrow \text{Im}(u) \rightarrow \text{Ker}(\psi) \rightarrow \text{Coker}(u) \rightarrow 0$  proves that  $\text{Ker}(\psi)$  is finitely generated since (by point (2) of Proposition B.13 it is an extension of finitely generated functors).

Analogously (ii) implies (i): let  $F$  be a finitely generated functor such that for any morphism  $\phi : G \rightarrow F$  from  $G$  finitely generated one has that  $\text{Ker}(\phi)$  is finitely generated. Given  $Q \twoheadrightarrow G$  a finitely generated subobject of  $F$  (hence  $Q \in \text{proj-}\mathcal{E}$  and  $G \hookrightarrow F$ ) we get a morphism  $Q \rightarrow F$  whose kernel is finitely generated and



so  $G$  is finitely presented which proves that  $F$  is coherent. On the other side if  $F$  is coherent for any morphism  $\phi : G \rightarrow F$  from  $G$  finitely generated we have that  $\text{Im}(\phi)$  is a finitely generated sub-object of  $F$  and hence it is finitely presented and so by (2)  $\ker(\phi)$  is finitely generated too.  $\square$

**Proposition B.15.** *The category  $\text{coh-}\mathcal{E}$  is closed under extension in  $\text{Mod-}\mathcal{E}$ . Moreover  $\text{coh-}\mathcal{E}$  is an abelian category and the canonical functor  $\text{coh-}\mathcal{E} \rightarrow \text{Mod-}\mathcal{E}$  is exact.*

*Proof.* Let  $0 \rightarrow F_1 \xrightarrow{\alpha} F \xrightarrow{\beta} F_2 \rightarrow 0$  be a short exact sequence in  $\text{Mod-}\mathcal{E}$  such that  $F_1, F_2 \in \text{coh-}\mathcal{E}$ . Hence by point (3) of Proposition B.13 we get that  $F$  is finitely presented (because it is an extension of finitely presented functors). Now let  $j : G \hookrightarrow F$  with  $G$  finitely generated. The commutative diagram with exact rows:

$$\begin{array}{ccccccc} 0 & \longrightarrow & F_1 \times_F G & \longrightarrow & G & \longrightarrow & \text{Im}(\beta j) \longrightarrow 0 \\ & & \downarrow & & \downarrow j & & \downarrow \\ 0 & \longrightarrow & F_1 & \xrightarrow{\alpha} & F & \xrightarrow{\beta} & F_2 \longrightarrow 0 \end{array}$$

proves that  $\text{Im}(\beta j)$  is finitely generated since it is a quotient of  $G$  which is finitely generated by hypothesis hence it is finitely presented too since  $F_2$  is coherent. Point (2) of Proposition B.14 proves that  $F_1 \times_F G$  is finitely generated (since  $\text{Im}(\beta j)$  is finitely presented and  $G$  is finitely generated) and so it is finitely presented too since  $F_1$  is coherent. This implies that  $G$  is finitely presented (applying point (3) of Proposition B.13).

Now, it remains to prove that given  $F \xrightarrow{f} G$  a morphism in  $\text{coh-}\mathcal{E}$  its kernel and cokernel in  $\text{Mod-}\mathcal{E}$  are coherent functors which proves that  $\text{coh-}\mathcal{E}$  is an abelian category and the inclusion functor  $\text{coh-}\mathcal{E} \rightarrow \text{Mod-}\mathcal{E}$  is exact. So let  $F \xrightarrow{f} G$  a morphism between two coherent functors. The image  $\text{Im}(f)$  of  $f$  is a finitely generated submodule of  $G$  and hence it is coherent by Definition B.6. Moreover the short exact sequence  $0 \rightarrow \text{Ker}(f) \rightarrow F \rightarrow \text{Im}(f) \rightarrow 0$  proves that  $\text{Ker}(f)$  is finitely generated (by point (ii) of Proposition B.14 since  $\text{Im}(f)$  is coherent and  $F$  finitely generated) and hence  $\text{Ker}(f)$  is coherent since it is a finitely generated sub-functor of the coherent functor  $F$ . Now, since  $G \twoheadrightarrow \text{Coker}(f)$  we deduce that  $\text{Coker}(f)$  is finitely generated and moreover let  $H$  be a finitely generated functor with  $\phi : H \rightarrow \text{Coker}(f)$  and let us consider the following commutative diagram with exact rows and columns:

$$\begin{array}{ccccccc} 0 & \longrightarrow & \text{Im}(f) & \hookrightarrow & G & \twoheadrightarrow & \text{Coker}(f) \longrightarrow 0 \\ & & \parallel & & \uparrow \psi & & \uparrow \phi \\ 0 & \longrightarrow & \text{Im}(f) & \hookrightarrow & G \times_{\text{Coker}(f)} H & \xrightarrow{q} & H \longrightarrow 0 \\ & & & & \uparrow & & \uparrow \\ & & & & \text{Ker}(\psi) & \xrightarrow{\cong} & \text{Ker}(\phi) \end{array}$$

$H$  and  $\text{Im}(f)$  are finitely generated hence  $G \times_{\text{Coker}(f)} H$  is finitely generated too (by point (2) of Proposition B.13). Hence  $\text{Ker}(\psi) = \text{Ker}(\phi)$  is finitely generated too by point (ii) of Proposition B.14 since  $G$  is coherent and so  $\text{Coker}(f)$  is coherent too.  $\square$

Let us recall that  $\text{Mod}(\mathcal{E}^\circ) = \mathcal{E}\text{-Mod}$  hence, passing to the opposite category, one can recover the previous results for left modules.

Given an additive category  $\mathcal{C}$ , Freyd introduced in [Fre66, page 99] the notion of *weak kernel* which permits to define the notion of *weak pull-back square*. An additive category  $\mathcal{C}$  admits weak pull back square if and only if it admits weak kernels. In [Nee01, Ch. 6, 6.1.1] Neeman independently introduced the notion of *homotopy pull-back square* which coincides with Freyd weak pull-back square.

**Definition B.16.** Let  $\mathcal{C}$  be an additive category and  $f : A \rightarrow B$  in  $\mathcal{C}$ . A weak kernel of  $f$  is a map  $i : K \rightarrow A$  such that  $f \circ i = 0$  and for any  $j : X \rightarrow A$  such that  $f \circ j = 0$  there exists, possibly many,  $\alpha : X \rightarrow K$  such that  $i \circ \alpha = j$ . An additive category  $\mathcal{C}$  has weakly (or equivalently homotopy) pull-back squares if given any pair  $f_i : X_i \rightarrow Y$  with  $i = 1, 2$  there exists an object  $Z$  with the dashed arrows such that any commutative diagram of this type can be completed with (a not necessarily unique) dotted arrow:

$$(5) \quad \begin{array}{ccccc} W & \xrightarrow{\quad \quad} & Z & \xrightarrow{g_1} & X_1 \\ & \searrow & \downarrow g_2 & & \downarrow f_1 \\ & & X_2 & \xrightarrow{f_2} & Y \end{array}$$

Passing throughout the opposite category one obtain the dual notion of weak cokernel and weak push-out.

**Proposition B.17.** ([Bel00, Proposition 4.5]). *Let  $\mathcal{P}$  be a projectively complete category. The following are equivalent:*

- (1)  $\mathcal{P}$  is right (respectively left) coherent;
- (2)  $\mathcal{P}$  admits weak kernels (respectively weak cokernels);
- (3) the Freyd category  $\text{fp-}\mathcal{P} = \text{coh-}\mathcal{P}$  (respectively  $\mathcal{P}\text{-fp} = \mathcal{P}\text{-coh}$ ) is an abelian exact full subcategory of  $\text{Mod-}\mathcal{P}$  (respectively  $\mathcal{P}\text{-Mod}$ ) whose projective (respectively injective) objects are exactly the representable functors in  $\mathcal{P}$ .

Moreover:

- $\mathcal{P}$  has kernels iff  $\text{fp-}\mathcal{P} = \text{coh-}\mathcal{P}$  is abelian with global dimension (i.e., the sup of the projective dimension of coherent functors)  $\text{gl.dim}(\text{coh-}\mathcal{P}) \leq 2$ ;
- $\text{fp-}\mathcal{P} = \text{coh-}\mathcal{P}$  is abelian with  $\text{gl.dim}(\text{coh-}\mathcal{P}) = 0$  iff  $\mathcal{P}$  is abelian semisimple and hence  $\mathcal{P} \cong \text{coh-}\mathcal{P}$ ;
- the  $\text{gl.dim}(\text{coh-}\mathcal{P}) = 1$  iff  $\mathcal{P}$  is not abelian semisimple but for any morphism  $f$  in  $\mathcal{P}$  we have that  $\text{Ker}(f)$  is split monic.

## APPENDIX C. $t$ -STRUCTURES

**C.1. Horthogonal classes.** Let  $\mathcal{C}$  be a pre-additive category and  $\mathcal{U}$  a full subcategory of  $\mathcal{C}$ ; we will denote by  $\mathcal{U}^\perp = \{C \in \mathcal{C} \mid \mathcal{C}(U, C) = 0 \ \forall U \in \mathcal{U}\}$  and by  ${}^\perp\mathcal{U} = \{C \in \mathcal{C} \mid \mathcal{C}(C, U) = 0 \ \forall U \in \mathcal{U}\}$ .

**C.2.  $t$ -structures.** The notion of  $t$ -structure is the analog for triangulated categories of that of torsion pair for abelian categories. Given  $\mathcal{C}$  a triangulated category, we will denote by  $[1]$  its suspension functor, by  $[n]$  its  $n^{\text{th}}$ -iterated functor with  $n \in \mathbb{Z}$  and we will use the notations  $X \rightarrow Y \rightarrow Z \xrightarrow{+1}$  or  $X \rightarrow Y \rightarrow Z \rightarrow X[1]$  for a distinguished triangle. When we say that  $\mathcal{U}$  is a subcategory of  $\mathcal{C}$ , we always mean

that  $\mathcal{U}$  is a full subcategory which is closed under isomorphisms, finite direct sums and direct summands.

If  $\mathcal{U}, \mathcal{V}$  are full subcategories of  $\mathcal{C}$ , then we denote by  $\mathcal{U} \star \mathcal{V}$  the category of extensions of  $\mathcal{V}$  by  $\mathcal{U}$ , that is, the full subcategory of  $\mathcal{C}$  consisting of objects  $X$  which may be included in a distinguished triangle  $U \rightarrow X \rightarrow V \xrightarrow{+}$  in  $\mathcal{C}$ , with  $U \in \mathcal{U}$  and  $V \in \mathcal{V}$ . As stated in [IY08] by the octahedral axiom we have  $(\mathcal{U} \star \mathcal{V}) \star \mathcal{W} = \mathcal{U} \star (\mathcal{V} \star \mathcal{W})$ . The subcategory  $\mathcal{U}$  is called *extension closed* if  $\mathcal{U} \star \mathcal{U} = \mathcal{U}$ . We note that if both  $\mathcal{U}$  and  $\mathcal{V}$  are extension closed hence  $\mathcal{U} \star \mathcal{V}$  is extension closed too. In general  $\mathcal{U} \star \mathcal{V}$  is not idempotently complete but by [IY08, Proposition 2.1] if the subcategories are orthogonal  $\mathcal{C}(\mathcal{U}, \mathcal{V}) = 0$  hence  $\mathcal{U} \star \mathcal{V}$  is closed under direct summands.

Let  $\mathcal{C}$  be a triangulated category, following [BBD82] a *t-structure* in  $\mathcal{C}$  is a pair  $\mathcal{D} := (\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0})$  of full subcategories of  $\mathcal{C}$  such that, setting  $\mathcal{D}^{\leq n} := \mathcal{D}^{\leq 0}[-n]$  and  $\mathcal{D}^{\geq n} := \mathcal{D}^{\geq 0}[n]$ , one has:

- (i)  $\mathcal{D}^{\leq 0} \subseteq \mathcal{D}^{\leq 1}$  and  $\mathcal{D}^{\geq 0} \supseteq \mathcal{D}^{\geq 1}$ ;
- (ii)  $\text{Hom}_{\mathcal{C}}(X, Y) = 0$ , for every  $X$  in  $\mathcal{D}^{\leq 0}$  and every  $Y$  in  $\mathcal{D}^{\geq 1}$ ;
- (iii) For any object  $X \in \mathcal{C}$  there exists a distinguished triangle in  $\mathcal{C}$

$$A \rightarrow X \rightarrow B \rightarrow A[1]$$

with  $A \in \mathcal{D}^{\leq 0}$  and  $B \in \mathcal{D}^{\geq 1}$  so  $\mathcal{C} = \mathcal{D}^{\leq 0} \star \mathcal{D}^{\geq 1}$ .

By [BBD82, Proposition 1.3.3, Theorem 1.3.6] the inclusion of  $\mathcal{D}^{\leq n}$  in  $\mathcal{C}$  (respectively the inclusion of  $\mathcal{D}^{\geq n}$  in  $\mathcal{C}$ ) has a right adjoint  $\delta^{\leq n}$  (respectively a left adjoint  $\delta^{\geq n}$ ) called the truncation functor providing for every object  $X$  in  $\mathcal{C}$  a unique morphism  $d: \delta^{\geq 1}(X) \rightarrow \delta^{\leq 0}(X)[1]$  such that the triangle

$$\delta^{\leq 0}(X) \rightarrow X \rightarrow \delta^{\geq 1}(X) \xrightarrow{d} \delta^{\leq 0}(X)[1]$$

is distinguished. This triangle is (up to a unique isomorphism) the unique distinguished triangle  $(A, X, B)$  with  $A$  in  $\mathcal{D}^{\leq 0}$  and  $B$  in  $\mathcal{D}^{\geq 1}$  and it is called the *approximating triangle* of  $X$  (for the *t-structure*  $\mathcal{D}$ ). The classes  $\mathcal{D}^{\leq 0}$  and  $\mathcal{D}^{\geq 0}$  are called the *aisle* and the *co-aisle* of the *t-structure*  $\mathcal{D}$ . More generally as proved in [KV88] (see also [Kel07, 7.2]) the data of a *t-structure* is equivalent to the datum of its aisle (which is by definition a full subcategory of  $\mathcal{U} \hookrightarrow \mathcal{C}$  closed by  $[1]$ , stable under extension and such that the inclusion functor admits a right adjoint) or of its co-aisle.

The category  $\mathcal{H}_{\mathcal{D}} := \mathcal{D}^{\leq 0} \cap \mathcal{D}^{\geq 0}$  is abelian and is called the heart of the *t-structure*. Moreover the truncation functors induce functors  $H_{\mathcal{D}}^i: \mathcal{C} \rightarrow \mathcal{H}_{\mathcal{D}}$ ,  $i \in \mathbb{Z}$ , called the *t-cohomological functors* associated with the *t-structure*  $\mathcal{D}$ , defined as follows:  $H_{\mathcal{D}}^0(X) := \delta^{\geq 0} \delta^{\leq 0}(X) \simeq \delta^{\leq 0} \delta^{\geq 0}(X)$  and for every  $i \in \mathbb{Z}$ ,  $H_{\mathcal{D}}^i(X) := H_{\mathcal{D}}^0(X[i])$ .

**C.3. Notation** Given  $\mathcal{D}$  a *t-structure* on a triangulated category  $\mathcal{C}$  we will denote by  $\mathcal{D}^{[a,b]} = \mathcal{D}^{\geq a} \cap \mathcal{D}^{\leq b} \subseteq \mathcal{C}$  with  $a \leq b$  in  $\mathbb{Z}$ . Hence  $\mathcal{D}^{[a,a]} = \mathcal{H}_{\mathcal{D}}[-a]$ .

Let  $\mathcal{A}$  be an abelian category, we use the notation  $X^{\bullet} := [X^i \rightarrow X^{i+1} \rightarrow \dots \rightarrow X^{i+n}]$  with  $n \in \mathbb{N}$  to indicate a complex in  $C(\mathcal{A})$  in degrees  $i$  to  $i+n$  whose remains terms (and arrows) are 0. We note by  $X^{\geq n}$  (respectively  $X^{\leq n}$ ) the complex which coincides with  $X^{\bullet}$  in degrees greater than (respectively less than) or equal to  $n$  and is zero otherwise.

**Definition C.4.** Given an abelian category  $\mathcal{A}$ , its (unbounded) derived category  $D(\mathcal{A})$  is a triangulated category which admits a  $t$ -structure, called the *natural  $t$ -structure*, whose aisle  $D(\mathcal{A})^{\leq 0}$  (respectively co-aisle  $D(\mathcal{A})^{\geq 0}$ ) is the subcategory of complexes without cohomology in positive (respectively negative) degrees.

**C.5.** Let  $\mathcal{P}$  be a projectively closed category. We will use the notation  $\cdots \rightarrow L \rightarrow \dot{M} \rightarrow N \rightarrow \cdots$  to indicate a complex in  $K(\mathcal{P})$  whose element  $M$  is placed in degree zero.

**C.6.** Following [BBD82] notation we will denote by  $\mathrm{Hom}_{\mathcal{C}}^n(X, Y) := \mathrm{Hom}_{\mathcal{C}}(X, Y[n])$ . Let us recall that any distinguished triangle  $X \rightarrow Y \rightarrow Z \xrightarrow{+}$  provides for any object  $T$  in  $\mathcal{C}$  the following long exact sequence (see [BBD82, page 18])

$$\begin{aligned} \cdots \rightarrow \mathrm{Hom}_{\mathcal{C}}^{-1}(T, Z) \rightarrow \mathrm{Hom}_{\mathcal{C}}(T, X) \rightarrow \mathrm{Hom}_{\mathcal{C}}(T, Y) \rightarrow \mathrm{Hom}_{\mathcal{C}}(T, Z) \rightarrow \mathrm{Hom}_{\mathcal{C}}^1(T, X) \rightarrow \cdots \\ \cdots \rightarrow \mathrm{Hom}_{\mathcal{C}}^{-1}(X, T) \rightarrow \mathrm{Hom}_{\mathcal{C}}(Z, T) \rightarrow \mathrm{Hom}_{\mathcal{C}}(Y, T) \rightarrow \mathrm{Hom}_{\mathcal{C}}(X, T) \rightarrow \mathrm{Hom}_{\mathcal{C}}^1(Z, T) \rightarrow \cdots \end{aligned}$$

**Definition C.7.** ([Kra07, 4.5 and 4.6]). Let  $\mathcal{C}$  be a triangulated category. A non-empty full subcategory  $\mathcal{N}$  of  $\mathcal{C}$  is called a *thick triangulated subcategory* if

- (TS1): for any  $X \in \mathcal{N}$  and  $i \in \mathbb{Z}$  we have  $X[i] \in \mathcal{N}$ ;
- (TS2): given any distinguished triangle  $X \rightarrow Y \rightarrow Z \xrightarrow{+}$  in  $\mathcal{C}$  if two objects from  $\{X, Y, Z\}$  belong to  $\mathcal{N}$ , then also the third one is in  $\mathcal{N}$ ;
- (TS3):  $\mathcal{N}$  is closed under direct factors.

One can attach to any thick subcategory  $\mathcal{N}$  of  $\mathcal{C}$  its multiplicative system (compatible with the triangulation)  $\Sigma(\mathcal{N})$  containing all the morphisms  $X \xrightarrow{f} Y$  in  $\mathcal{C}$  fitting in a distinguished triangle  $X \xrightarrow{f} Y \rightarrow Z \xrightarrow{+}$  with  $Z \in \mathcal{N}$ . Hence one can perform the quotient category  $\mathcal{C}/\mathcal{N} := \mathcal{C}[\Sigma(\mathcal{N})^{-1}]$  (which is a category in a wider sense since it could be not locally small i.e.; the homomorphisms between two objects in the quotient does not form a set in general) endowed with the quotient functor  $Q : \mathcal{C} \rightarrow \mathcal{C}/\mathcal{N}$  such that by [Kra07, Proposition 4.6.2]:

- (1) the category  $\mathcal{C}/\mathcal{N}$  carries a unique triangulated structure such that  $Q$  is exact;
- (2) a morphism in  $\mathcal{C}$  is annihilated by  $Q$  if and only if it factors through an object in  $\mathcal{N}$  and moreover  $\mathcal{N} = \mathrm{Ker} Q$  (since it is thick);
- (3) every exact functor  $\mathcal{C} \rightarrow \mathcal{U}$  annihilating  $\mathcal{N}$  factors uniquely through  $Q$  via an exact functor  $\mathcal{C}/\mathcal{N} \rightarrow \mathcal{U}$ .

**C.8.** Given  $\mathcal{A}$  an abelian category the subcategory

$$\mathcal{N} := \{X^\bullet \in K(\mathcal{A}) \mid H^i(X^\bullet) = 0, \forall i \in \mathbb{Z}\}$$

is a thick subcategory of  $K(\mathcal{A})$  and the quotient  $K(\mathcal{A})/\mathcal{N} =: D(\mathcal{A})$  defines the derived category of  $\mathcal{A}$  (which might be non-locally small).

**Remark C.9.** Let  $\mathcal{A}$  be an abelian category. Let us recall that if  $\mathcal{A}$  has a generating family  $\mathcal{P}$  of projectives (see Definition B.8) hence

$$\mathcal{N} := \{X^\bullet \in K(\mathcal{A}) \mid \mathrm{Hom}_{K(\mathcal{A})}(P[i], X^\bullet) = 0, \forall i \in \mathbb{Z} \text{ and } \forall P \in \mathcal{P}\}.$$

Dually if  $\mathcal{A}$  has a cogenerating family of injectives  $\mathcal{I}$

$$\mathcal{N} := \{X^\bullet \in K(\mathcal{A}) \mid \mathrm{Hom}_{K(\mathcal{A})}(X^\bullet, I[i]) = 0, \forall i \in \mathbb{Z} \text{ and } \forall I \in \mathcal{I}\}.$$

**Lemma C.10.** (See [Kra07, Exercise(5.1.5)]). Let  $\mathcal{A}$  be an abelian category with enough projectives and finite global dimension  $\mathrm{gl.dim}(\mathcal{A}) = n$  (i.e.; any object in  $\mathcal{A}$  has a projective resolution of length less than or equal to  $n$ ). Let denote by  $\mathcal{P}$  the projective objects in  $\mathcal{A}$ . Hence the null system  $\mathcal{N} \cap K(\mathcal{P}) = \{0\}$  coincide with the zero complex and

$$K(\mathcal{P}) \cong D(\mathcal{A}).$$

Dually if  $\mathcal{A}$  is an abelian category with enough injectives and finite global injective dimension  $\mathrm{inj.gl.dim}(\mathcal{A}) = n$ . Let denote by  $\mathcal{I}$  the projective objects in  $\mathcal{A}$ . Hence  $\mathcal{N} \cap K(\mathcal{I}) = \{0\}$

$$K(\mathcal{I}) \cong D(\mathcal{A})$$

and  $\mathcal{N} \cap K(\mathcal{P}) = \{0\}$ .

A generalized version of this Lemma is the following Proposition due to Kashiwara and Schapira in [KS06, Proposition 13.2.6]:

**Proposition C.11.** Let  $\mathcal{C}$  be an abelian category,  $\mathcal{I}$  a full additive subcategory of  $\mathcal{C}$  such that:

- (1)  $\mathcal{I}$  is cogenerating;
- (2) there exists  $d > 0$  such that any exact sequence  $Y_d \rightarrow \cdots \rightarrow Y_1 \rightarrow Y \rightarrow 0$  with  $Y_j \in \mathcal{I}$  for  $1 \leq j \leq d$  we have  $Y \in \mathcal{I}$ . Hence the canonical functor

$$\frac{K(\mathcal{I})}{K(\mathcal{I}) \cap \mathcal{N}} \xrightarrow{\cong} D(\mathcal{C})$$

is a triangulated equivalence of categories (where  $\mathcal{N}$  is the null system of acyclic complexes in  $D(\mathcal{C})$ ).

The following Lemma of Schneiders [Sch99, Lemma 1.2.17] provides a compatibility condition between a  $t$ -structure  $\mathcal{T}$  on  $\mathcal{C}$  and a null system in order to induce a new  $t$ -structure on the quotient  $\mathcal{C}/\mathcal{N}$ :

**Lemma C.12.** [Sch99, Lemma 1.2.17] Given a  $t$ -structure  $(\mathcal{T}^{\leq 0}, \mathcal{T}^{\geq 0})$  on a triangulated category  $\mathcal{C}$  and a saturated null system (or equivalently a thick full triangulated subcategory)  $\mathcal{N}$  with  $Q : \mathcal{C} \rightarrow \mathcal{C}/\mathcal{N}$  its canonical quotient functor; the essential images  $(Q(\mathcal{T}^{\leq 0}), Q(\mathcal{T}^{\geq 0}))$  form a  $t$ -structure on  $\mathcal{C}/\mathcal{N}$  if and only if for any distinguished triangle  $X_1 \rightarrow X_0 \rightarrow N \xrightarrow{+1}$  with  $X_1 \in \mathcal{T}^{\geq 1}$ ,  $X_0 \in \mathcal{T}^{\leq 0}$  and  $N \in \mathcal{N}$  we have  $X_1, X_0 \in \mathcal{N}$ .

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DIPARTIMENTO DI MATEMATICA, UNIVERSITÀ DEGLI STUDI DI PADOVA, VIA TRIESTE 63, I-35121  
PADOVA ITALY

*E-mail address:* `fiorot@math.unipd.it`